Comparison of Cardinalities

A class function $\underline{F} : \underline{X} \to \underline{Y}$ is *bijective* if it is injective and surjective. (one-to-one onto)

Lemma The following are equivalent for $\underline{F} : \underline{X} \to \underline{Y}$

- $\underline{F}$ is a bijection
- There exists a (necessarily unique) $\underline{F}^{-1} : \underline{Y} \to \underline{X}$ such that:

  $\underline{F}^{-1} \circ \underline{F} = id_{\underline{X}}$ $\quad$ $\underline{F} \circ \underline{F}^{-1} = id_{\underline{Y}}$

Def X, Y **have the same cardinality** (are **in bijective correspondence**, are **equipotent**) if there exists a bijection $f: X \to Y$.
(Notation: $|X| = |Y|$)

Lemma

- $|X| = |X|$
- $|X| = |Y| \Rightarrow |Y| = |X|$
- $|X| = |Y|$ & $|Y| = |Z| \Rightarrow |X| = |Z|$

Def X has cardinality less than or equal to that of Y if there is an injection $i : X \to Y$.
(Notation $|X| \leq |Y|$)

Lemma

- $|X| \leq |X|$
- $|X| \leq |Y|$ & $|Y| \leq |Z| \Rightarrow |X| \leq |Z|$.
- $|X| \leq |Y|$ & $|Y| \leq |X| \Rightarrow |X| = |Y|$.

Proof of last is nontrivial. So we separate as a theorem.

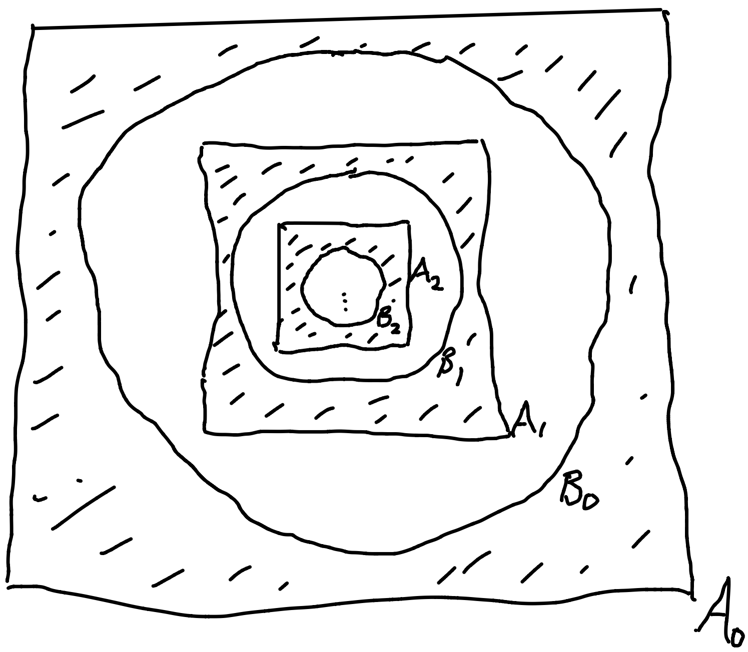
Theorem (Schröder–Bernstein) If $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$.

Lemma Given sets $A_1 \subseteq B_0 \subseteq A_0$ with $|A_1| = |A_0|$ then $|B_0| = |A_0|$.

Proof Let $f : A_0 \to A_1$ be a bijection

So $A_1 = f[A_0]$.

Define $B_{n+1} = f[B_n]$; $A_{n+1} = f[A_n]$



A_2
B_2
B_1
A_1
B_0
A_0

By induction $A_{n+1} \subseteq B_n \subseteq A_n$ for all n

Define

$$C_n := A_n - B_n$$

$$C := \bigcup_{n \geq 0} C_n$$

$$D := A_0 - C$$

Note that $f[C_n] = f[A_n - B_n]$
$= f[A_n] - f[B_n] = A_{n+1} - B_{n+1} = C_{n+1}$

So $f[C] = f[\bigcup_{n\geq 0} C_n] = \bigcup_{n\geq 0} f[C_n] = \bigcup_{n\geq 0} C_{n+1}$

N.B. $f[C] \cup D = B_0$.

Define $g : A_0 \to B_0$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in D \end{cases}$$

Both $g|_C$ and $g|_D$ are injective
Their images are disjoint. So g is injective

So g is a bijection from A_0 onto its image

$$g[A_0] = g[C \cup D] = g[C] \cup g[D]$$

$$= f[C] \cup D = B_0 .$$

$\square$

Proof of S-B theorem.

Suppose $i : X \to Y$ and $j : Y \to X$ are injections

Then $j[i[X]] \subseteq j[Y] \subseteq X$

Moreover $|X| = |j[i[X]]|$

By the lemma $|X| = |j[Y]| = |Y|$. $\square$

Def We say that the cardinality of X is <u>strictly less than</u> that of Y if $|X| \leq |Y|$ and $|X| \neq |Y|$.
(Notation $|X| < |Y|$.)

Lemma

- $|X| \nless |X|$
- $|X| < |Y|$ & $|Y| < |Z| \Rightarrow |X| < |Z|$. (Use S-B theorem.)
- $|X| < |Y| \Rightarrow |X| \leq |Y|$.

Def We say that the cardinality of X is <u>very strictly less than</u> that of Y if
$|X| \leq |Y|$ and there is no surjection from X to Y.
(Notation $|X| \ll |Y|$.)

Lemma

- $|X| \ll |Y| \Rightarrow |X| < |Y|$. (Easy exercise)
- $|X'| \leq |X| \ll |Y| \leq |Y'| \Rightarrow |X'| \ll |Y'|$.
  (Exercise.)

Infinite sets

Def I is *infinite* if there exists $I' \subset I$ such that $|I'| = |I|$.

Proposition T.f.a.e

(1) I is infinite

(2) There exists a proper injection $S: I \to I$

(3) $|\mathbb{N}| \leq |I|$

Proof

(1) ⇒ (2) Let $f : I \to I'$ be a bijection from I to $I' \subset I$

Then $I \xrightarrow{f} I' \xrightarrow{\subset} I$ is the required proper injection

$(2) \Rightarrow (3)$

Because s is not injective there exist $z \in I - \text{image}(s)$

Define $f : \mathbb{N} \to I$ by recursion:

$$f(0) = z$$
$$f(n+1) = s(f(n))$$

Then f is injective. $\boxed{\text{Exercise}}$

$(3) \Rightarrow (1)$ Given $f: \mathbb{N} \to I$ injective

define $J = f[\mathbb{N}]$ $K = I - f[\mathbb{N}]$

Then $I = J \uplus K$

Define $I' = f[\mathbb{N}^+] \uplus K$

$\mathbb{N} = \{0, 1, \cdots\}$

$\mathbb{N}^+ = \{1, 2, \ldots\}$

I' is a proper subset of I : $f(0) \in I - I'$

The function $g: I \to I'$ below is a bijection

$$g(x) = \begin{cases} f(n+1) & \text{if } x = f(n) \\ x & \text{if } x \in K \end{cases}$$

$\square$

Def X is countable (denumerable) if $X = \emptyset$
or there exists a surjection $e : \mathbb{N} \to X$

Proposition If X is countable then either $|X| = |[n)|$
where $[n) ::= \{x \in \mathbb{N} \mid 0 \leq x < n\}$ for some $n \in \mathbb{N}$
or $|X| = |\mathbb{N}|$.

Proof If $X = \emptyset$ then $|X| = |[0)|$.
Otherwise there is a surjection $e : \mathbb{N} \to X$.

Define $f : \mathbb{N} \longrightarrow X \mathbin{\dot\cup} \{*\}$ by:

$$f(n) = \begin{cases} \text{the } (n+1)^{th} \text{ distinct element of } X \text{ in the sequence} \\ \quad e(0), e(1), e(2), e(3), \dots \quad \text{if this exists} \\ * \quad \text{otherwise} \end{cases}$$

If there exists n s.t. $f(n) = *$ then let m be the smallest n for which $f(n) = *$

$$f\restriction_{[m)} : [m) \to X \text{ is a bijection}$$

Otherwise $f : \mathbb{N} \to X$ is a bijection

$\square$

Theorem $|X| < |\mathcal{P}(X)|$.

Proof (Cantor) $x \mapsto \{x\}$ is an injection from X to $\mathcal{P}(X)$.

Suppose for contradiction that $e : X \to \mathcal{P}X$ is surjective

Consider the set

$$Y := \{x \in X \mid x \notin e(x)\}$$

Since e is surjective $Y = e(y)$ for some $y \in X$

Then

$$y \in Y \iff y \notin e(y) \iff y \notin Y.$$

$\rightarrow\leftarrow$

$\square$

Corollary $\mathcal{P}\mathbb{N}$ is uncountable.

<u>Proposition</u> $|\mathbb{N}| = |\mathbb{Q}|$

<u>Lemma</u> There is an injection $\mathbb{R} \to \mathcal{P}\mathbb{Q}$

<u>Proof</u> $x \mapsto \{q \mid q < x\}$ $\square$

<u>Lemma</u> There is an injection $\mathcal{P}\mathbb{N} \to \mathbb{R}$

<u>Proof</u> $X \mapsto \sum_{n=0}^{\infty} \frac{1_X(n)}{3^{n+1}}$ where $1_X(n) = \begin{cases} 1 & \text{if } n \in X \\ 0 & \text{otherwise} \end{cases}$ $\square$

<u>Theorem</u> $|\mathbb{R}| = |\mathcal{P}\mathbb{N}|$

<u>Corollary</u> $\mathbb{R}$ is uncountable.