Cardinal Arithmetic

$|\underline{X}| = |\underline{Y}| \qquad |\underline{X}| \leq |\underline{Y}| \qquad |\underline{X}| < |\underline{Y}| \qquad |\underline{X}| \ll |\underline{Y}|$

- If $\underline{X}$ is infinite $|\mathbb{N}| \leq |\underline{X}|$
- $|\underline{X}| \ll |\wp \underline{X}|$ (Last week)

The powerclass operation

$$\underline{X} \longmapsto \wp \underline{X}$$

induces a corresponding operation on class functions. Given

$$F : \underline{X} \rightsquigarrow \underline{Y}$$

we define

$$\wp F : \wp \underline{X} \to \wp \underline{Y}$$

by

$$(\wp F)(Z) := F[Z] \qquad (Z \subseteq \underline{X})$$

__Proposition__ The operation $F \mapsto \wp F$ is __functorial__; i.e.,

- For any $\underline{X}$, we have $\wp(id_{\underline{X}}) = id_{(\wp \underline{X})}$
- For $F : \underline{X} \to \underline{Y}$ and $G : \underline{Y} \to \underline{Z}$, we have

$$\wp(G \circ F) = (\wp G) \circ (\wp F) : \wp \underline{X} \to \wp \underline{Z}$$

Proposition The operation $X \mapsto \wp X$ preserves cardinalities; i.e.,

$$|X| = |Y| \Rightarrow |\wp X| = |\wp Y| \qquad (1)$$

It is also monotone

$$|X| \leq |Y| \Rightarrow |\wp X| \leq |\wp Y| \qquad (2)$$

Proof

(1) follows from functoriality because:

If $|X| = |Y|$ then we have $f : X \to Y$ and $f^{-1} : Y \to X$ s.t. $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$. Then

$$\wp(f^{-1}) \circ (\wp f) = \wp(f^{-1} \circ f) = \wp(id_X) = id_{\wp X}$$

and

$$(\wp f) \circ \wp(f^{-1}) = \ldots = id_{\wp Y}$$

So indeed $|\wp X| = |\wp Y|$.

(2) We use functoriality to establish:

$$1 \leq |X| \leq |Y| \Rightarrow |\mathcal{P}X| \leq |\mathcal{P}Y| \qquad (2)'$$

Lemma Suppose X is nonempty. Then $f : X \to Y$ is an injection iff it has a left inverse; i.e. there exists $g : Y \to X$ s.t. $g \circ f = id_X$

Proof $\Leftarrow$ Exercise.

$\Rightarrow$ Suppose X is nonempty and $f : X \to Y$ is injective. Define $g : Y \to X$ by:

$$g(y) = \begin{cases} \text{the unique } x \in X \text{ s.t. } f(x) = y & \text{if } y \in \text{image}(f) \\ x_0 & \text{if } y \notin \text{image}(f) \end{cases}$$

where x_0 is an arbitrary element of X.

It is immediate that $g \circ f = id_X$. $\square$

To prove ②'.

Suppose $1 \leq |X| \leq |Y|$

So there is some injection $f: X \to Y$

By the lemma there exists $g: Y \to X$ with $g \circ f = id_X$. Then

$$(\wp g) \circ (\wp f) = \wp(g \circ f) = \wp(id_X) = id_{\wp X}$$

So $\wp f : \wp X \to \wp Y$ has a left inverse. Clearly $\wp X$ is nonempty.

By the lemma $\wp f$ is an injection. So $|\wp X| \leq |\wp Y|$.

This proves ②'.

For ② we just need to consider the additional case $X = \emptyset$. Then

$$\wp X = \wp \emptyset = \{\emptyset\} \subseteq \wp Y$$

The inclusion function $\wp X \subseteq \wp Y$ is trivially injective.

So indeed $|X| \leq |Y|$. I.e. ② holds.

$\square$

In lecture 1 we defined

$\underline{X} \times \underline{Y}$ product

$\underline{Y}^{X}$ exponential

We now define

$\underline{X} + \underline{Y}$ sum

by:

$$\underline{X} + \underline{Y} := \{(0,x) \mid x \in \underline{X}\} \cup \{(1,y) \mid y \in \underline{Y}\}$$

where $0 = \emptyset$ and $1 = \{\emptyset\}$ (e.g.)

<u>Proposition</u> If X, Y are sets then so are $X+Y$, $X \times Y$, Y^X

Given $f: X \to X'$ and $g: Y \to Y'$, define

$$f+g : X+Y \to X'+Y'$$

$$z \longmapsto \begin{cases} (0, f(x)) & \text{if } z = (0, x) \\ (1, g(y)) & \text{if } z = (1, y) \end{cases}$$

$$f \times g : X \times Y \to X' \times Y'$$

$$(x, y) \mapsto (f(x), g(y))$$

$$g^f : Y^{(X')} \longrightarrow (Y')^X$$

$$h \mapsto (x \mapsto g(h(f(x))))$$

$$(h \mapsto g \circ h \circ f)$$

Note that g^f reverses the direction of f.

Proposition The operations above are functorial; i.e. they preserve identity and composition.

(N.B. the operation g^f is contravariant in f. All other cases are covariant.)

Proposition If $|X| = |X'|$ and $|Y| = |Y'|$ then:

- $|X+Y| = |X'+Y'|$
- $|X \times Y| = |X' \times Y'|$
- $|Y^X| = |(Y')^{(X')}|$

If $|X| \leq |X'|$ and $|Y| \leq |Y'|$ then:

- $|X+Y| \leq |X'+Y'|$
- $|X \times Y| \leq |X' \times Y'|$
- $|Y^X| \leq |(Y')^{(X')}|$ except in the case that $X = Y' = \emptyset \neq X'$

Cardinal Arithmetic (Part 2)

Notation

$|X| + |Y|$ means $|X + Y|$

$|X| \cdot |Y|$ means $|X \times Y|$

$|Y|^{|X|}$ means $|Y^X|$

0 means $|\emptyset|$

1 means $|\{\emptyset\}|$

2 means $|\{\emptyset, \{\emptyset\}\}|$

$\aleph_0$ means $|\mathbb{N}|$

Arithmetical laws for cardinalities

- $|X| + 0 = |X|$ $\quad$ $\big(|X| + |\emptyset|$
  $|X + \emptyset| = |X|\big)$
- $|X| + |Y| = |Y| + |X|$
- $|X| + (|Y| + |Z|) = (|X| + |Y|) + |Z|$

- $|X| \cdot 1 = |X|$
- $|X| \cdot |Y| = |Y| \cdot |X|$
- $|X| \cdot (|Y| \cdot |Z|) = (|X| \cdot |Y|) \cdot |Z|$

- $|X| \cdot 0 = 0$
- $|X| \cdot (|Y| + |Z|) = |X| \cdot |Y| + |X| \cdot |Z|$

- $|X|^0 = 1$
- $|X|^{|Y|+|Z|} = (|X|^{|Y|}) \cdot (|X|^{|Z|})$

—

- $1^{|X|} = 1$
- $(|Y| \cdot |Z|)^{|X|} = (|Y|^{|X|}) \cdot (|Z|^{|X|})$

—

- $|X|^1 = |X|$
- $|X|^{|Y| \cdot |Z|} = (|X|^{|Y|})^{|Z|}$ $\quad (*)$

Example proof of $(*)$

we need to give a bijection from $X^{Y \times Z}$ to $(X^Y)^Z$

Such a bijection is $\Lambda : X^{Y \times Z} \to (X^Y)^Z$ defined by:

$$\underset{X^{Y \times Z}}{\Lambda(\underset{\cap}{g})} := (\underset{\underset{Z}{\cap}}{z} \mapsto (\underset{\underset{Y}{\cap}}{y} \mapsto g(y,z)))$$

<u>Exercise</u> define $\Lambda^{-1} : (X^Y)^Z \to X^{Y \times Z}$

and show it is an inverse to Λ

Therefore $|X^{Y \times Z}| = |(X^Y)^Z|$, i.e. $|X|^{|Y| \cdot |Z|} = (|X|^{|Y|})^{|Z|}$.

(cf. "Currying" in computer science.)

The above algebraic laws say that cardinalities form an <u>exponential semiring</u>

<u>Proposition</u> $|\mathcal{P}X| = 2^{|X|}$

i.e., $|\mathcal{P}X| = |\{\emptyset, \{\emptyset\}\}^X|$

<u>Proof</u> Exercise. $\square$

The above laws are familiar from ordinary arithmetic

A major way in which cardinal arithmetic differs from ordinary arithmetic is that cancellation laws <u>fail</u>

e.g, in ordinary arithmetic we have:

$$x + z = y + z \Rightarrow x = y$$

$$x \cdot z = y \cdot z \Rightarrow x = y \quad \text{provided that } z \neq 0$$

For example

$$0 + |\mathbb{N}| = |\mathbb{N}| = |\mathbb{N}| + |\mathbb{N}| \quad \text{but} \quad 0 \neq |\mathbb{N}|$$

$$1 \cdot |\mathbb{N}| = |\mathbb{N}| = |\mathbb{N}| \cdot |\mathbb{N}| \quad \text{but} \quad 1 \neq |\mathbb{N}|$$

This relies on two important equalities

$$|\mathbb{N}| = |\mathbb{N}| + |\mathbb{N}| \qquad \aleph_0 = \aleph_0 + \aleph_0$$

$$|\mathbb{N}| = |\mathbb{N}| \cdot |\mathbb{N}| \qquad \aleph_0 = \aleph_0 \cdot \aleph_0$$

We say that $|X|$ is its <u>own square</u> if

$$|X| = |X| \cdot |X|$$

<u>Lemma</u> If $|X| \geq 2$ and $|X|$ is its own square then

- X is infinite, and
- $|X| = |X| + |X|$.

Proof Easily $|X| \leq |X| + |X|$

For the converse

$$|X| + |X| = 2 \cdot |X| \leq |X| \cdot |X| = |X|$$

By Schröder-Bernstein $|X| = |X| + |X|$

Let $f : X + X \to X$ be some bijection

The $f[\{(0, x) \mid x \in X\}]$ is a proper subset of X that is in bijection with X. Hence X is infinite. $\square$

Proposition If $2 \leq |X|$ and $|X|$ is its own square then $2^{|X|}$ is also its own square.

Proof $2^{|X|} \cdot 2^{|X|} = 2^{|X| + |X|} = 2^{|X|}$. $\square$

Theorem If $|X|$ is its own square and $Y \subseteq X$ is such that $|Y| \ll |X|$ then $|X - Y| = |X|$.

Proof Consider the composite function

$$Y \xrightarrow{\subseteq} X \xrightarrow{f} X \times X \xrightarrow{(x_1, x_2) \mapsto x_1} X$$

where $f : X \to X \times X$ is a bijection.

Because $|Y| \ll |X|$ the above function is not a surjection.

So there exists $x_0 \in X$ not in the image

i.e., for every $y \in Y$, we have $f(y) = (x_1, x_2)$ where $x_1 \neq x_0$

We therefore have a function

$$X \xrightarrow{x \mapsto (x_0, x)} X \times X - f[Y]$$

which is obviously injective.

Hence $|X| \leq |X \times X - f[Y]| \overset{f \text{ is a bijection}}{=} |X - Y|$

Trivially $|X - Y| \leq |X|$. So indeed $|X| = |X - Y|$ by S-B. $\square$