A recap on orders

Strict partial order

irreflexivity: $x \not< x$

transitivity: $x<y \wedge y<z \Rightarrow x<z$

(it follows that $\neg(x<y \wedge y<x)$)

non-strict partial order

reflexivity: $x \leq x$

transitivity: $x\leq y \wedge y \leq z \Rightarrow x \leq z$

antisymmetry: $x \leq y \wedge y \leq x \Rightarrow x=y$

Strict and non-strict orders are interdefinable:

$$x<y \Leftrightarrow x \leq y \wedge x \neq y$$

$$x \leq y \Leftrightarrow x<y \vee x=y$$

A <u>total order</u> (also <u>linear order</u>) is a partial order +

trichotomy: $x<y \vee x=y \vee y<x$ | totality: $x \leq y \vee y \leq x$

Suppose $\leq$ is a partial order on a class $\underline{X}$

For a subclass $\underline{A} \subseteq \underline{X}$ we define

$$\underline{A} \leq x \Leftrightarrow \forall a \in \underline{A}\ \ a \leq x \quad (x \text{ is an } \underline{\text{upper bound}} \text{ of (for) } \underline{A})$$

$$\underline{A} < x \Leftrightarrow \forall a \in \underline{A}\ \ a < x \quad (\text{a } \underline{\text{strict upper bound}})$$

The <u>Supremum</u> (<u>least upper bound</u>) of $\underline{A}$ is an element $\sup \underline{A} \in \underline{X}$ s.t.

$$\underline{A} \leq \sup \underline{A} \quad \text{and} \quad \forall x \in \underline{X}\ (\underline{A} \leq x \Rightarrow \sup \underline{A} \leq x)$$

The Supremum need not exist. If it exists it is unique.

Recap continued

For $x \in \underline{X}$ we define

$\downarrow x := \{ y \in \underline{X} \mid y < x \}$ <u>strict down-closure</u>

$\Downarrow x := \{ y \in \underline{X} \mid y \leq x \}$ <u>(non-strict) down-closure</u>

For $\underline{A} \subseteq \underline{X}$ we similarly define

$\downarrow \underline{A} := \{ y \in \underline{X} \mid \exists x \in \underline{A} \ \ y < x \}$ <u>strict down-closure</u>

$\Downarrow \underline{A} := \{ y \in \underline{X} \mid \exists x \in \underline{A} \ \ y \leq x \}$ <u>(non-strict) down-closure</u>

Ordinals

Last weeks: $|X| = |Y|$ etc.

$0, 1, 2, 3, 4, 5, \ldots$

$\omega, \omega+1, \omega+2, \omega+3, \omega+4, \ldots$

$\omega\cdot 2, \omega\cdot 2+1, \omega\cdot 2+2, \omega\cdot 2+3, \ldots$ $\quad (\omega\cdot 2 = \omega+\omega)$

$\omega\cdot 3, \omega\cdot 3+1, \omega\cdot 3+2, \omega\cdot 3+3, \ldots$

$\omega\cdot 4, \ldots$

$\omega\cdot 5, \ldots$

$\vdots$

$\omega^2, \omega^2+1, \omega^2+2, \ldots$ $\quad (\omega^2 = \omega\cdot\omega)$

$\omega^2+\omega, \omega^2+\omega+1, \ldots$

$\omega^2+2\omega, \ldots$

$\vdots$

$\omega^2\cdot 2 \ldots$

$\vdots$

$\omega^2\cdot 3 \ldots$

$\vdots$

$\omega^3 \ldots$ $\quad (\omega^3 = \omega^2\cdot\omega)$

$\vdots$

ω^4

$\vdots$

ω^5

$\vdots$

ω^ω

$\vdots$

ω^{ω^ω} $\quad \omega^{(\omega^\omega)}$

$\vdots$

$\omega^{\omega^{\omega^\omega}}$

$\vdots$

ϵ_0, (the smallest ordinal greater than all of $\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots$)

$\epsilon_0+1, \epsilon_0+2, \ldots$

An <u>Ordinal structure</u> is given by a class $\underline{Ord}$, a partial order $\leq$ on $\underline{Ord}$, an element $0 \in \underline{Ord}$, class functions $S : \underline{Ord} \to \underline{Ord}$, $\mathrm{Sup} : \mathcal{P}(\underline{Ord}) \to \underline{Ord}$ satisfying:

- 0 is the least element of $\underline{Ord}$

  $\forall \alpha \in \underline{Ord} \quad 0 \leq \alpha$

- S is a <u>successor operation</u>

  $\alpha < S(\alpha)$ and $\forall \beta \in \underline{Ord} \quad \alpha < \beta \Rightarrow S(\alpha) \leq \beta$

- Sup is a <u>supremum operation</u> on $\underline{Ord}$, i.e.,

  $\forall$ subsets $A \subseteq \underline{Ord} \quad A \leq \mathrm{Sup}\, A$

  and $\forall \beta \in \underline{Ord} \quad A \leq \beta \Rightarrow \mathrm{Sup}\, A \leq \beta$

- The principle of <u>transfinite induction</u> (T.I.) holds, i.e.,

  If $\underline{X} \subseteq \underline{Ord}$ is such that

  - $0 \in \underline{X}$ (Base case)
  - $\alpha \in \underline{X} \Rightarrow S(\alpha) \in \underline{X}$ (Step case)
  - For every subset $A \subseteq \underline{X}$, we have $\sup A \in \underline{X}$ (Limit case)

  Then $\underline{X} = \underline{Ord}$.

$\boxed{\text{Notation}}$ To ease readability we write

$\alpha + 1$ instead of $S(\alpha)$

lower case — $\sup A$ instead of $\mathrm{Sup}\, A$ — upper case

Also given a set I and a family $(\alpha_i)_{i \in I}$ (given by $\alpha_{(-)} : I \to \underline{Ord}$)

we write $\sup\limits_{i \in I} \alpha_i$ for $\sup\{\alpha_i \mid i \in I\}$

Proposition $\leq$ is a linear order on $\underline{\mathrm{Ord}}$

Proof Let $\beta \in \underline{\mathrm{Ord}}$ be arbitrary.

We prove $\alpha \leq \beta$ or $\beta \leq \alpha$ for all $\alpha \in \underline{\mathrm{Ord}}$, by T.I. on α

Base case $\alpha = 0$. Trivially $0 \leq \beta$.

Step case Suppose $\alpha \leq \beta$ or $\beta \leq \alpha$ (induction hypothesis)

R.t.p. $\alpha+1 \leq \beta$ or $\beta \leq \alpha+1$.

In the case that $\underline{\alpha \leq \beta}$:

either $\alpha = \beta$ or $\alpha+1 \leq \beta$ because $(-)+1$ is a successor.

so $\beta \leq \alpha+1$ or $\alpha+1 \leq \beta$ as required.

In the case that $\underline{\beta \leq \alpha}$: we have $\beta \leq \alpha < \alpha+1$. So $\beta \leq \alpha+1$.

Limit case

Suppose A is a set of ordinals satisfying

$$\forall \alpha \in A \; (\alpha \leq \beta \text{ or } \beta \leq \alpha) \quad (*)$$

R.t.p. $\sup A \leq \beta$ or $\beta \leq \sup A$

By $(*)$ either $(\forall \alpha \in A \;\; \alpha \leq \beta)$

or $(\exists \alpha \in A \;\; \beta \leq \alpha)$

(This uses classical logic)

In the first case $A \leq \beta$ hence $\sup A \leq \beta$.

In the second case $\beta \leq \alpha \leq \sup A$.

So indeed $\alpha \leq \beta$ or $\beta \leq \alpha$ $\forall \alpha\, \beta$ as required $\square$

Proposition For every ordinal α, the class $\downarrow\alpha$ is a set.

Proof Transfinite induction on α.

Base case $\alpha = 0$ $\quad \downarrow 0 = \emptyset$ which is a set.

Step case Suppose $\downarrow\alpha$ is a set.

Then $\downarrow(\alpha+1) = (\downarrow\alpha) \cup \{\alpha\}$, which is a set.

Limit case Suppose $A \subseteq \underline{\mathrm{Ord}}$ is a subset s.t. $\downarrow\alpha$ is a set for all $\alpha \in A$.

Then $\downarrow(\sup A) = \bigcup\{\downarrow\alpha \mid \alpha \in A\}$, which is a set.

$\square$

Theorem $\underline{\mathrm{Ord}}$ is a proper class.

Proof If $\underline{\mathrm{Ord}}$ were a set, we could define

$$\alpha := \sup \underline{\mathrm{Ord}} \in \underline{\mathrm{Ord}}$$

, hence $\alpha + 1 \in \underline{\mathrm{Ord}}$

$$\alpha+1 \leq \sup \underline{\mathrm{Ord}} = \alpha < \alpha + 1$$

a contradiction!

$\square$

The ordinals separate into two kinds

$\alpha \in \underline{\mathrm{Ord}}$ is called a <u>successor ordinal</u> if there exists $\alpha' \in \underline{\mathrm{Ord}}$ with $\alpha = \alpha' + 1$.

$\alpha \in \underline{\mathrm{Ord}}$ is called a <u>limit ordinal</u> if there exists a subset $A \subseteq {\downarrow}\alpha$ with $\alpha = \sup A$ (equivalently if $\alpha = \sup {\downarrow}\alpha$)

<u>Proposition</u> Every ordinal is either a successor ordinal or a limit ordinal (but not both)

<u>Proof</u> Again transfinite induction. Exercise. $\square$

Two reformulations of transfinite induction.

Transfinite induction (successor / limit version)

If $\underline{X} \subseteq \underline{Ord}$ satisfied

(base) • $0 \in \underline{X}$

(step) • $\alpha \in \underline{X} \Rightarrow \alpha + 1 \in \underline{X}$

(limit) • $\downarrow\lambda \subseteq \underline{X} \Rightarrow \lambda \in \underline{X}$ where $\lambda > 0$ is a limit ordinal

(if $\alpha \in \underline{X}$ for all $\alpha < \lambda$ then $\lambda \in X$)

Then $\underline{X} = \underline{Ord}$.

Transfinite induction (uniform version)

If $\underline{X} \subseteq \underline{Ord}$ satisfies

- $\downarrow\alpha \subseteq \underline{X} \Rightarrow \alpha \in \underline{X}$ for all ordinals α

(if $\beta \in \underline{X}$ for all $\beta < \alpha$ then $\alpha \in \underline{X}$)

Then $\underline{X} = \underline{Ord}$.

The class $\underline{Ord}$ has $0 \in \underline{Ord}$ and $S : \underline{Ord} \to \underline{Ord}$

So by the recursion theorem for $\mathbb{N}$ there is $i : \mathbb{N} \to \underline{Ord}$

$$i(0) = 0 \qquad \underbrace{i(n+1)}_{\text{in } \mathbb{N}} = \underbrace{i(n) + 1}_{\text{in } \underline{Ord}} \qquad (\text{i.e } S(i(n)))$$

$$\omega := \sup_{n \in \mathbb{N}} i(n)$$

$\vdots$

Transfinite recursion theorem

Let $\underline{A}$ be a class with $a \in \underline{A}$ and class functions $F : \underline{A} \to \underline{A}$ and $L : \mathcal{P}\underline{A} \to \underline{A}$. There is a unique class function

$$T : \underline{Ord} \to \underline{A}$$

Satisfying

$$T(0) = a$$

$$T(\alpha+1) = F(T(\alpha))$$

$$T(\lambda) = L(\{T(\alpha) \mid \alpha < \lambda\}) \qquad \lambda > 0 \text{ a limit ordinal}$$

Corollary Any two ordinal structures are isomorphic.

Exercise formulate and prove — adapt Lecture 2.

Outline proof of TRT

An *approximation* is a function $t : \downarrow\gamma \to \underline{A}$ for some $\gamma \in \underline{Ord}$ that satisfies

$$0 < \gamma \Rightarrow t(0) = a$$
$$\alpha + 1 < \gamma \Rightarrow t(\alpha+1) = F(t(\alpha))$$
$$\lambda < \gamma \Rightarrow t(\lambda) = L(\{t(\alpha) \mid \alpha < \lambda\}) \quad \lambda > 0 \text{ a limit ordinal}$$

By T.I. (successor / limit version on α we prove)

⊛ For every $\alpha \in \underline{Ord}$ there exists a unique approximation $t_\alpha : \downarrow\alpha \to \underline{A}$.

For every α define:

$$T(\alpha) = t_{\alpha+1}(\alpha)$$

This satisfies the required conditions.
For uniqueness, suppose $T' : \underline{Ord} \to \underline{A}$ satisfies the conditions then we prove $T'(\alpha) = T(\alpha)$ for all α, by T.I (succ/lim version)

$\square$

Parameterised transfinite recursion theorem

Suppose $\underline{A}$ and $\underline{Z}$ are classes with class functions

$B : \underline{Z} \to \underline{A}$, $F : \underline{Z} \times \underline{A} \to \underline{A}$ and $L : \underline{Z} \times \mathcal{P}\underline{A} \to \underline{A}$

There is a unique class function $T : \underline{Z} \times \underline{Ord} \to \underline{A}$ satisfying:

$$T(z, 0) = B(z)$$

$$T(z, \alpha+1) = F(z, T(z, \alpha))$$

$$T(z, \lambda) = L(z, \{T(z, \alpha) \mid \alpha < \lambda\}) \quad \lambda > 0 \text{ a limit ordinal}$$

Using the parameterised transfinite recursion theorem we define $T : \underline{Ord} \times \underline{Ord} \to \underline{Ord}$ using

$$B(\alpha) = \alpha$$

$$F(\alpha, \beta) = \beta + 1$$

$$L(\alpha, A) = \sup A$$

Then $T : \underline{Ord} \times \underline{Ord} \to \underline{Ord}$ is the unique class function s.t.

$$T(\alpha, 0) = \alpha$$

$$T(\alpha, \beta+1) = F(\alpha, T(\alpha,\beta)) = T(\alpha,\beta) + 1$$

$$T(\alpha, \lambda) = L(\alpha, \{T(\alpha,\beta) \mid \beta < \lambda\}) = \sup \{T(\alpha,\beta) \mid \beta < \lambda\}$$

We have defined <u>ordinal addition</u> $\alpha + \beta$ as the unique class function $(\cdot) + (\cdot) : \underline{Ord} \times \underline{Ord} \to \underline{Ord}$ satisfying

$$\alpha + 0 = \alpha$$

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1 \qquad \left(\alpha + S(\beta) = S(\alpha + \beta)\right)$$

+ being defined

S on $\underline{Ord}$

+ being defined

S on $\underline{Ord}$

$$\alpha + \lambda = \sup_{\beta < \lambda} \alpha + \beta$$

+ just defined

It follows that $\alpha + 1 = \underline{S}(\alpha)$

<u>Exercise:</u> $1 + \omega = \omega \neq \omega + 1$

cf. cardinal arithmetic

$1 + \aleph_0 = \aleph_0 = \aleph_0 + 1$

Ordinal Addition is <u>not</u> commutative

Similarly define ordinal multiplication $(\cdot)\cdot(\cdot) : \underline{Ord} \times \underline{Ord} \to \underline{Ord}$

$$\alpha \cdot 0 = 0$$
$$\alpha \cdot (\beta+1) = \alpha \cdot \beta + \alpha$$
$$\alpha \cdot \lambda = \sup_{\beta<\lambda} \alpha \cdot \beta \qquad \lambda>0 \text{ a limit ordinal}$$

Exercise: $2 \cdot \omega = \omega \neq \omega + \omega = \omega \cdot 2$

ordinal multiplication is <u>not</u> commutative

Ordinal exponentiation $(\cdot)^{(\cdot)} : \underline{Ord} \times \underline{Ord} \to \underline{Ord}$

$$\alpha^0 = 1$$
$$\alpha^{(\beta+1)} = \alpha^\beta \cdot \alpha$$
$$\alpha^\lambda = \sup_{\beta<\lambda} \alpha^\beta \qquad \lambda>0 \text{ is a limit ordinal}$$

Exercise $\omega^2 = \omega \cdot \omega \neq \omega$ cf. $\aleph_0^2 = \aleph_0 \cdot \aleph_0 = \aleph_0$

$\omega = 2^\omega \neq \omega^\omega$ cf. $\aleph_0 \neq 2^{\aleph_0} = \aleph_0^{\aleph_0}$

Construction of an ordinal structure

Recall the von Neumann numbers

$$0 = \emptyset \qquad = \emptyset$$
$$1 = \{\emptyset\} \qquad = \{0\}$$
$$\vdots$$
$$n+1 = n \cup \{n\} \qquad = \{0, 1, \cdots, n\}$$
$$\vdots$$
$$\omega = \{0, 1, \cdots, n, \cdots\} \quad (= \mathrm{v}\mathbb{N})$$
$$\omega+1 = \{0, 1, \cdots, n, \cdots, \omega\}$$

A von Neumann ordinal is the set of all strictly smaller von Neumann ordinals.

$$0 = \emptyset$$
$$\alpha+1 = \alpha \cup \{\alpha\} =: \alpha^+$$
$$\sup A = \bigcup A$$

A set of sets X is a <u>von Neumann ordinal</u> (<u>vN ordinal</u>) if for every $Y \subseteq X$ satisfying: (i) $\emptyset \in X \Rightarrow \emptyset \in Y$; (ii) $\forall z \in Y \; (z^+ \in X \Rightarrow z^+ \in Y)$ and (iii) $\forall W \subseteq Y \; (\bigcup W \in X \Rightarrow \bigcup W \in Y)$ it holds that $Y = X$.

<u>Theorem</u> The class $\underline{\mathrm{vNord}}$ (of all vN ordinals) together with $\emptyset$, $(\cdot)^+ : \underline{\mathrm{vNord}} \to \underline{\mathrm{vNord}}$, $\bigcup : \mathcal{P}\underline{\mathrm{vNord}} \to \underline{\mathrm{vNord}}$ is an ordinal structure. $\square$