PDE šolsko leto 2021/22

1. Reši toplotno enačbo
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
ob pogojih
$$u(0,t) = 0, \quad u(\pi,t) = 0, \qquad \text{za } t > 0,$$
in
$$u(x,0) = \begin{cases} \frac{2x}{\pi} & : \quad 0 < x < \frac{\pi}{2} \\ \sin x & : \quad \frac{\pi}{2} \leq x < \pi \end{cases} \qquad \text{za } 0 < x < \pi,$$

**Rešitev:**

Vpeljemo
$$u(x,t) = X(x)T(t)$$
od koder dobimo
$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$
Iz robnih pogojev $u(0,t) = 0, \quad u(\pi,t) = 0, \quad$ za $t > 0$, sledi
$$X(0) = 0, \quad X(\pi) = 0.$$

Torej netrivialne rešitve dobimo v primeru $\lambda > 0$ in tedaj ob upoštevanju rbnih pogojev dobimo
$$\lambda = n^2, \quad n \in \mathbb{N}$$
$$X_n(x) = \sin nx, \quad n \in \mathbb{N}.$$

Rešimo
$$\frac{T'}{T} = -n^2.$$
$$T(t) = Ae^{-n^2 t}.$$

Pišimo
$$T_n(t) = A_n e^{-n^2 t}.$$
$$u_n(x,t) = T_n(t)X_n(x) = A_n e^{-n^2 t} \sin nx \quad n \in \mathbb{N},$$
$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin nx.$$

Upoštevajmo začetni pogoj
$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx, \quad x \in (0,\pi).$$
$$A_n = b_n = \frac{2}{\pi} \int_0^{\pi} u(x,0) \sin(nx) \, \mathrm{d}x =$$

Naj bo $n > 1$.

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \frac{2x}{\pi}\sin(nx)\ \mathrm{d}x + \int_0^{\frac{\pi}{2}} \sin x \sin(nx)\ \mathrm{d}x\right) =$$

$$\frac{4}{\pi^2}\left(\int_0^{\frac{\pi}{2}} x\sin(nx)\ \mathrm{d}x + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin x \sin(nx)\ \mathrm{d}x\right) =$$

$$= \frac{4}{\pi^2}\left[\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{2}{2\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{4}{\pi^2}\left[\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right] + \frac{1}{\pi}\left[-\frac{\sin(1-n)\frac{\pi}{2}}{1-n} + \frac{\sin(1+n)\frac{\pi}{2}}{1+n}\right]$$

Posebej moramo obravnavati primere, ko $n = 1$ in v primeru, ko je $n > 1$ in je $n$ sod oziroma lih. Obravnavajmo najperj primer $n = 2k$, $k \in \mathbb{N}$, $k > 1$.

$$b_{2k} == \frac{4}{\pi^2}\left[\frac{-\pi}{4k}\cos\frac{2k\pi}{2} + \frac{1}{(2k)^2}\sin\frac{2k\pi}{2}\right] + \frac{1}{\pi}\left[-\frac{\sin(1-2k)\frac{\pi}{2}}{1-2k} + \frac{\sin(1+2k)\frac{\pi}{2}}{1+2k}\right] =$$

$$= \frac{4}{\pi^2}\left[\frac{-\pi}{4k}\cos\frac{2k\pi}{2} + \frac{1}{(2k)^2}\sin\frac{2k\pi}{2}\right] + \frac{1}{\pi}\left[\frac{\sin(2k-1)\frac{\pi}{2}}{2k-1} + \frac{\sin(1+2k)\frac{\pi}{2}}{1+2k}\right] =$$

$$= \frac{4}{\pi^2}\left[\frac{-\pi}{4k}(-1)^k\right] + \frac{1}{\pi}\left[\frac{(-1)^{k+1}}{1-2k} + \frac{(-1)^k}{1+2k}\right] =$$

$$= \frac{1}{\pi}(-1)^{k+1}\left[\frac{1}{k} + \frac{(4k)}{1-4k^2}\right] =$$

$$= \frac{1}{\pi}\frac{(-1)^k}{(4k^2-1)k}.$$

Naj bo $n = 2k+1$, $k \in \mathbb{N}$.

$$A_{2k+1} = b_{2k+1} == \frac{4}{\pi^2}\left[\frac{-\pi}{2(2k+1)}\cos\frac{(2k+1)\pi}{2} + \frac{1}{(2k+1)^2}\sin\frac{(2k+1)\pi}{2}\right] + \frac{1}{\pi}\left[-\frac{\sin(1-(2k+1))\frac{\pi}{2}}{1-(2k+1)} +\right.$$

$$= \frac{4}{\pi^2}\frac{1}{(2k+1)^2}\sin\frac{(2k+1)\pi}{2} = \frac{4}{\pi^2}\frac{(-1)^k}{(2k+1)^2}.$$

Izračunajmo še

$$A_1 = b_1 = \frac{4}{\pi^2}\left(\int_0^{\frac{\pi}{2}} x\sin(x)\ \mathrm{d}x + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin x \sin(x)\ \mathrm{d}x\right) =$$

$$= \frac{4}{\pi^2}\left[-x\cos x + \sin x\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(1-\cos 2x)\ \mathrm{d}x =$$

$$= \frac{4}{\pi^2}\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{-\pi}{2}\right) + \frac{1}{\pi}\left(x + \frac{\sin 2x}{2}\right)\Bigg|_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{4}{\pi^2} + \frac{1}{\pi}\frac{\pi}{2} = \frac{4}{\pi^2} + \frac{1}{2}.$$

$$u(x,t) = \left(\frac{1}{2} + \frac{4}{\pi^2}\right) e^{-t} \sin x + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{4k^3 - k} e^{-4k^2 t} \sin(2kx) +$$

$$+ \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^2} e^{-(2k+1)^2 t} \sin((2k+1)\,x)$$

2. Reši toplotno enačbo

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

ob pogojih

$$u(0,t) = 0, \quad u(\pi,t) = 0, \qquad \text{za } t > 0,$$

in

$$u(x,0) = x\,(\pi - x) \qquad \text{za } 0 < x < \pi.$$

**Rešitev:**

Vpeljemo

$$u(x,t) = X(x)T(t)$$

od koder dobimo

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

Iz robnih pogojev $u(0,t) = 0, \quad u(\pi,t) = 0, \qquad \text{za } t > 0$, sledi

$$X(0) = 0, \quad X(\pi) = 0.$$

Torej netrivialne rešitve dobimo v primeru $\lambda > 0$ in tedaj ob upoštevanju rbnih pogojev dobimo

$$\lambda = n^2, \quad n \in \mathbb{N}$$

$$X_n(x) = \sin nx, \quad n \in \mathbb{N}.$$

Rešimo

$$\frac{T'}{T} = -n^2.$$

$$T(t) = Ae^{-n^2 t}.$$

Pišimo

$$T_n(t) = A_n e^{-n^2 t}.$$

$$u_n(x,t) = T_n(t)X_n(x) = A_n e^{-n^2 t} \sin nx \quad n \in \mathbb{N},$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin nx.$$

Upoštevajmo začetni pogoj

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx, \quad x \in (0,\pi).$$

$$x\,(\pi - x)\sum_{n=1}^{\infty} A_n \sin nx, \quad x \in (0,\pi).$$

Razvijmo funkcijo $x\,(\pi - x)$ za $0 < x < \pi$. v sinusno Fourierovo vrsto. Naj bo $n \in \mathbb{N}$, $n \geq 1$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\,(\pi - x)\sin(nx)\,\mathrm{d}x = 2\int_0^{\pi} x\sin(nx)\,\mathrm{d}x - \frac{2}{\pi}\int_0^{\pi} x^2\sin(nx)\,\mathrm{d}x =$$

$$= 2\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^{\pi} - \frac{2}{\pi}\left[-\frac{x^2}{n}\cos(nx) + \frac{(2x)}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)\right]_0^{\pi} =$$

$$= -\frac{2\pi}{n}(-1)^n - \left(-\frac{2\pi}{n}(-1)^n + \frac{4}{\pi n^3}(-1)^n - \frac{4}{\pi n^3}\right) =$$

$$= -\frac{4}{\pi n^3}(-1)^n + \frac{4}{\pi n^3} =$$

$$= \begin{cases} 0 & : \ n \ = 2k \\ \frac{8}{(2k-1)^3\pi} & : \ n \ = 2k-1, \quad k \in \mathbb{N} \end{cases}$$

$$u(x,t) = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}e^{-(2k+1)^2 t}\sin((2k+1)x)$$

.

3. Poišči rešitev valovne enačbe

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

za $(x,t) \in [0,1] \times [0,\infty)$ ob pogojih

$$u(0,t) = 0, \quad u(1,t) = 0, \quad \text{za } t > 0,$$

in

$$u(x,0) = \sin 5\pi x + 2\sin 7\pi x, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \text{za } 0 < x < \pi.$$

Rešitev: $u(x,t) = \cos(5\pi ct)\sin(5\pi x) + 2\sin(7\pi ct)\sin(7\pi x)$.

4. Reši toplotno enačbo

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

ob pogojih

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad \text{za } t > 0,$$

in

$$u(x,0) = \begin{cases} x & : \ 0 < x < \frac{\pi}{2} \\ \pi - x & : \ \frac{\pi}{2} \leq x < \pi \end{cases}$$

**Rešitev:**

Vpeljemo

$$u(x,t) = X(x)T(t)$$

od koder dobimo

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

Iz robnih pogojev $u(0,t) = 0, \quad u(\pi,t) = 0, \quad$ za $t > 0$, sledi

$$X(0) = 0, \quad X(\pi) = 0.$$

Torej netrivialne rešitve dobimo v primeru $\lambda > 0$ in tedaj ob upoštevanju rbnih pogojev dobimo

$$\lambda = n^2, \quad n \in \mathbb{N}$$

$$X_n(x) = \sin nx, \quad n \in \mathbb{N}.$$

Rešimo

$$\frac{T'}{T} = -n^2.$$

$$T(t) = Ae^{-n^2 t}.$$

Pišimo

$$T_n(t) = A_n e^{-n^2 t}.$$

$$u_n(x,t) = T_n(t)X_n(x) = A_n e^{-n^2 t} \sin nx \quad n \in \mathbb{N},$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin nx.$$

Upoštevajmo začetni pogoj

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx, \quad x \in (0,\pi).$$

$$A_n = b_n = \frac{2}{\pi} \int_0^{\pi} u(x,0) \sin(nx) \, \mathrm{d}x =$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, \mathrm{d}x =$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} x \sin nx \, \mathrm{d}x + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} x \sin nx \, \mathrm{d}x - \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} x \sin nx \, \mathrm{d}x =$$

$$= \frac{-1}{n} \cos \frac{n\pi}{2} + \frac{2}{\pi n^2} \sin \frac{n\pi}{2} + -\frac{2}{n} \cos n\pi + \frac{2}{n} \cos \frac{n\pi}{2} -$$

$$\frac{2}{n} \cos n\pi + \frac{2}{n^2 \pi} \sin(n\pi) + \frac{1}{n} \cos \frac{n\pi}{2} - \frac{2}{\pi n^2} \sin \frac{n\pi}{2} =$$

$$= \frac{4}{\pi n^2} \sin \frac{n\pi}{2} =$$

$$\begin{cases} 0 & : \quad n = 2k \\ \frac{4}{\pi(2k-1)^2} \sin \frac{(2k-1)\pi}{2} & : \quad n = 2k-1 \end{cases}$$

$$u(x,t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin((2k-1)x) e^{-(2k-1)^2} t.$$

Upoštevamo

$$\sin\frac{n\pi}{2} = \begin{cases} (-1)^{k+1} & : \quad n = 2k-1 \\ 0 & : \quad n = 2k \end{cases}$$

$$\int x \sin nx \, \mathrm{d}x = -\frac{x}{n}\cos{(nx)} + \frac{1}{n^2}\sin nx \, \mathrm{d}x.$$

5. Poišči rešitev valovne enačbe

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

za $(x,t) \in [0,\pi] \times [0,\infty)$ ob pogojih

$$u(0,t) = 0, \quad u(\pi,t) = 0, \qquad \text{za } t > 0,$$

in

$$u(x,0) = \cos x, \quad \frac{\partial u}{\partial t}(x,0) = 1, \qquad \text{za } 0 < x < \pi.$$

**Rešitev:**

Vpeljemo

$$u(x,t) = X(x)T(t)$$

od koder dobimo

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = -\lambda.$$

Iz robnih pogojev $u(0,t) = 0, \quad u(\pi,t) = 0, \qquad \text{za } t > 0$, sledi

$$X(0) = 0, \quad X(\pi) = 0.$$

Torej netrivialne rešitve dobimo v primeru $\lambda > 0$ in tedaj ob upoštevanju rbnih pogojev dobimo

$$\lambda = n^2, \quad n \in \mathbb{N}$$

$$X_n(x) = \sin nx, \quad n \in \mathbb{N}.$$

Rešimo

$$\frac{1}{c^2}\frac{T''}{T} = -n^2.$$

$$T'' + c^2 n^2 T = 0,$$

$$T_n(t) = A_n \cos{(cnt)} + B_n \sin{(cnt)},$$

$$u_n(x,t) = T_n(t)X_n(x) = (A_n \cos{(cnt)} + B_n \sin{(cnt)}) \sin nx \quad n \in \mathbb{N},$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (A_n \cos{(cnt)} + B_n \sin{(cnt)}) \sin nx.$$

Upoštevajmo začetni pogoj

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx, \quad x \in (0,\pi).$$

$$u_t(x,t) = \sum_{n=1}^{\infty} (A_n cn(-\sin(cnt) + B_n cn \cos(cnt)) \sin nx,$$

$$u_t(x,0) = \sum_{n=1}^{\infty} (B_n cn) \sin nx.$$

Torej določiti moramo koeficiente naslednjih sinusnih Fourierovih vrst

$$\cos x = \sum_{n=1}^{\infty} A_n \sin nx, \quad x \in (0,\pi)$$

in

$$1 = \sum_{n=1}^{\infty} (B_n cn) \sin nx.$$

$$b_n = B_n cn = \frac{2}{\pi} \int_0^{\pi} \sin nx \; \mathrm{d}x =$$

$$= \frac{2}{\pi} \left( \frac{-\cos nx}{n} \right) \Bigg|_0^{\pi} =$$

$$= -\frac{2}{n\pi} (\cos n\pi - 1) = -\frac{2}{n\pi} ((-1)^n - 1) =$$

$$= \begin{cases} \frac{4}{n\pi} & : \quad n = 2k-1 \\ 0 & : \quad n = 2k \end{cases}$$

$$B_n == \begin{cases} \frac{4}{n^2 c\pi} & : \quad n = 2k-1 \\ 0 & : \quad n = 2k \end{cases}$$

Izračunajmo še koeficiente $A_n$, $n \in \mathbb{N}$.

$$A_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \; \mathrm{d}x.$$

Naj bo $n > 1$.

$$A_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \; \mathrm{d}x =$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} \left[\sin(nx + x) + \sin(nx - x)\right] \; \mathrm{d}x =$$

$$= -\frac{1}{\pi} \left[ \frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^{\pi} =$$

$$= -\frac{1}{\pi} \left[ \frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \left( \frac{1}{n+1} + \frac{1}{n-1} \right) \right] =$$

$$= -\frac{1}{\pi} \left[ \frac{(n-1)(-1)^{n+1} + (n+1)(-1)^{n-1}}{(n+1)(n-1)} - \frac{2n}{n^2-1} \right] =$$

$$= \begin{cases} 0 & : \quad n = 2k-1 \\ \frac{8k}{\pi(4k^2-1)} & : \quad n = 2k \end{cases}$$

izračunajmo še $A_1$.

$$A_1 = \frac{2}{\pi}\int_0^\pi \cos x \sin x \ \mathrm{d}x = \frac{1}{\pi}\int_0^\pi \sin 2x \ \mathrm{d}x = 0.$$

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_{2k}\cos 2kct + B_{2kct}\right)\sin 2kx +$$

$$+\sum_{k=1}^{\infty}\left(A_{2k-1}\cos(2k-1)ct + B_{(2k-1)ct}\right)\sin(2k-1)x =$$

$$= \sum_{k=1}^{\infty}\left(\frac{8k}{\pi\left(4k^2-1\right)}\cos(2kct)\sin(2kx) + \frac{4}{c\pi\left(2k-1\right)^2}\sin\left(\left(2k-1\right)ct\right)\sin\left(\left(2k-1\right)x\right)\right).$$