Lecture 6: Well-orders

Functions between partially ordered sets (recap)

Suppose X, Y are partially ordered sets and $f: X \to Y$ is a function

- We say f is monotone (increasing, order preserving) if $x \leq x' \Rightarrow f(x) \leq f(x')$
- We say f is strictly monotone (strictly increasing, etc.) if $x < x' \Rightarrow f(x) < f(x')$.
- If f is strictly monotone then it is monotone.
- If f is monotone and injective then it is strictly monotone.
- If X is a linear order then f is strictly monotone iff it is an order embedding:
- f is an (order) embedding if $x \leq x' \Leftrightarrow f(x) \leq f(x')$.

(• Every embedding is injective)

- f is an (order) isomorphism if it is monotone and there exists a (necessarily unique) monotone $f^{-1}: Y \to X$ such that $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$.

(• Every isomorphism is an embedding.)

- An (order) automorphism is an isomorphism from a partially ordered set to itself

Prologue : The least ordinal property

Principle of transfinite induction (uniform version):

If $\underline{X} \subseteq \underline{Ord}$ satisfies:

$\forall \alpha \in Ord$ if $(\forall \beta < \alpha \ \beta \in \underline{X})$ then $\alpha \in \underline{X}$

Then $\underline{X} = \underline{Ord}$.

Proposition (The least ordinal property)
If $\underline{X} \subseteq \underline{Ord}$ and $\underline{X}$ is nonempty then $\underline{X}$ has a smallest element (that is there exists $\alpha \in \underline{X}$ s.t. $\forall \beta \in \underline{X} \ \alpha \leq \beta$).

Proof Suppose $\underline{X} \subseteq \underline{Ord}$ and $\underline{X}$ does not have a least element. Then we show that $\underline{X} = \emptyset$.

Define $\underline{Y} = \underline{Ord} - \underline{X}$

We use T.I to show that $\underline{Y} = \underline{Ord}$ hence $\underline{X} = \emptyset$.

Suppose $\alpha \in \underline{Ord}$ is such that $\forall \beta < \alpha \ \beta \in \underline{Y}$.

If $\beta \notin \underline{Y}$ then β would be the smallest element of $\underline{X}$.
But $\underline{X}$ has no least element.

So $\beta \in \underline{Y}$.

By T.I. $\underline{Y} = \underline{Ord}$. Hence $\underline{X} = \emptyset$. $\square$

Proposition Consider the following 3 statements regarding a relation $R \subseteq X \times X$ (where, for convenience, X is a set)

1. For every subset $Z \subseteq X$ if

   - for all $x \in X$ if (for all $y \in X$, yRx implies $y \in Z$) then $x \in Z$ ⊛

   (yRx: $(y,x) \in R$)

   then $Z = X$.

2. For every nonempty subset $Z \subseteq X$, there exists an <u>R-minimal</u> element $z \in Z$ (i.e., $\forall z' \in Z$ we do not have $z' R z$).

3. There is no <u>R-decreasing</u> infinite sequence in X

   (An <u>R-decreasing</u> sequence is $x_{(-)} : \mathbb{N} \to X$ satisfying for all n, $x_{n+1} R x_n$.)

Then $(1) \Leftrightarrow (2) \Rightarrow (3)$

(Furthermore $(3) \Rightarrow (2)$ if we assume the axiom of dependent choice!) ← Leave to a future lecture

<u>Proof.</u> <u>$(1) \Rightarrow (2)$</u> similar to the proof above about ordinals.

<u>$(2) \Rightarrow (1)$</u> Suppose (2)

Also suppose $Z \subseteq X$ satisfies ⊛. We need to prove $Z = X$.

<u>Suppose not.</u> Then $Y := X - Z$ is nonempty.

So Y has an R-minimal element x (by 2).

So for all $y \in X$ we have $yRx \Rightarrow y \notin Y$ i.e. $y \in Z$.

By ⊛ $x \in Z$; i.e., $x \notin Y$. A contradiction.

Therefore indeed $Z = X$.

(2) $\Rightarrow$ (3) Suppose 2.

Suppose that $x_{(-)} : \mathbb{N} \to X$ is an R-decreasing sequence.

Then $\{x_n \mid n \in \mathbb{N}\} \subseteq X$ has no R-minimal element.

$\square$

Definition A relation $R \subseteq X \times X$ is said to be well-founded if property 1 (equivalently 2) of the proposition above hold.

Corollary of the proposition above

If R is a well-founded relation then it is acyclic, i.e., there is no finite sequence of elements $x_1, \dots, x_k \in X$ s.t. $x_1 R x_2 \quad x_2 R x_3 \quad \dots \quad x_k R x_1$. In particular R is

- Irreflexive $\forall x \; \neg\, x R x$
- Strictly antisymmetric $\forall x, y \quad x R y \Rightarrow \neg\, y R x$

Definition A well-ordered set (well-order) is given by a set X with a total order such that the strict order relation $<$ on X is well-founded.

Examples of well orders

For every ordinal α the set $\downarrow\alpha$ (remember $\downarrow\alpha := \{\beta \in \underline{\mathrm{Ord}} \mid \beta \leq \alpha\}$) is well-orded by $<$.

(Proof Suppose $X \subseteq \downarrow\alpha$ is non-empty.
By the least ordinal property X has a least element.
This is then a $<$-minimal element.
So $<$ on $\downarrow\alpha$ satisfies def 2 of a well-ordering.)

Temporary questions: • Can we find other examples of well-orders; i.e., ones not isomorphic to any $\downarrow\alpha$?

• Can $\downarrow\alpha$ and $\downarrow\beta$ be isomorphic if $\alpha \neq \beta$?

Two technical lemmas

Lemma A: If f is an embedding from a well-ordered set A to itself then f is *inflationary*; i.e, for all $x \in A$, $x \leq f(x)$.

Proof Suppose f is not inflationary.
Then the subset $B := \{x \in A \mid f(x) < x\}$ is non-empty.
So B has a least element x_0. So $f(x_0) < x_0$
By strict monotonicity $f(f(x_0)) < f(x_0)$.
That is $f(x_0) \in B$. This contradicts x_0 being the least element of B. $\square$

An *initial segment* of a totally ordered set X is just a down closed subset $I \subseteq X$; i.e. if $x \in I$ and $y \leq x$ then $y \in I$.
It is a *proper* initial segment if I is a proper subset of X.

Lemma B: (1) If I is a proper initial segment of a well-ordered set A then $I = \downarrow a$ for a unique $a \in A$.
(2) Similarly if a set I is an initial segment of the ordinals then $I = \downarrow \alpha$ for a unique ordinal α.

Proof (1) The set $I^c := A - I$ is nonempty.
Let a be the least element of I^c. Then $I = \downarrow a$. $\boxed{\text{Exercise}}$
(2) Similar. $\square$

We answer our temporary questions in the negative.

The classification theorem for well-orders

Every well-ordered set is order isomorphic to $\downarrow\alpha$ for a unique ordinal α.

We postpone the (lengthy) proof

Using the classification theorem, for any well-ordered set A we write $\mathrm{ord}(A)$ for the unique ordinal α s.t. $A \cong \downarrow\alpha$ is isomorphic to

(N.B. • strictly we should write $\mathrm{ord}(A, <_A)$)

We call $\mathrm{ord}(A)$ the <u>order type</u> of the well-order.

<u>Theorem</u> The following are equivalent for well-ordered sets A, B.

(1) $A \cong B$

(2) $\mathrm{ord}(A) = \mathrm{ord}(B)$

<u>Theorem</u> T.f.a.e. for well-ordered sets A, B

(1) A embeds in B

(2) A is isomorphic to an initial segment of B

(3) $\mathrm{ord}(A) \leq \mathrm{ord}(B)$

<u>Corollary</u> For well-ordered sets A, B:

(1) If A and B embed in each other, then $A \cong B$

(2) Either A embeds in B or B embeds in A.

Outline proof of second theorem above

(2) ⇒ (1). The embedding from A to B is

$$A \xrightarrow{\cong} I \xrightarrow{\subseteq} B .$$

(1) ⇒ (3) Suppose $i : A \to B$ is an embedding

Then we have an embedding

$$\downarrow \mathrm{Ord}(A) \xrightarrow{\cong} A \xrightarrow{i} B \xrightarrow{\cong} \downarrow \mathrm{Ord}(B)$$

In other words we have an embedding

$$\downarrow \alpha \xrightarrow{j} \downarrow \beta \qquad \text{where } \alpha = \mathrm{Ord}(A) \quad \beta = \mathrm{Ord}(B)$$

We need to show $\alpha \leq \beta$.

Suppose instead $\beta < \alpha$, then $j(\beta) \in \downarrow\beta$ so $j(\beta) < \beta$.

This contradicts Lemma A.

(3) ⇒ (2)

$$A \xrightarrow{\cong} \downarrow \mathrm{Ord}(A) \xrightarrow{\subseteq} \downarrow \mathrm{Ord}(B) \xrightarrow{\cong} B$$

Let I be the image of this composite.

I is an initial segment of B.

And the above map is an isomorphism from A to I.

$\square$

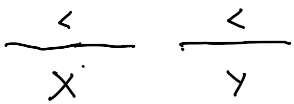
Constructions on well-ordered sets

Suppose $(X, <_X)$ and $(Y, <_Y)$ are well-ordered sets

Define:

$$(X, <_X) + (Y, <_Y) := (X+Y, <_{X+Y}) \quad \text{where:}$$

$$z <_{X+Y} w \iff \exists x, x' \; z = (0,x) \wedge w = (0,x') \wedge x < x'$$
$$\text{or } \exists x, y \; z = (0,x) \wedge w = (1,y)$$
$$\text{or } \exists y y' \; z = (1,y) \wedge w = (1,y') \wedge y < y'$$

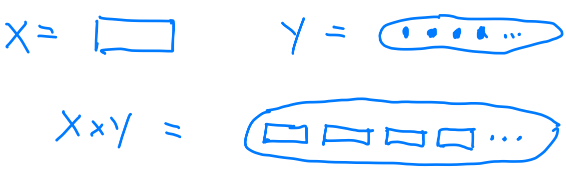
(Idea )

$$(X, <_X) \times (Y, <_Y) := (X \times Y, <_{X \times Y}) \quad \text{where:}$$

$$(x,y) <_{X\times Y} (x',y') \iff y < y' \text{ or}$$
$$y = y' \text{ and } x < x'.$$

(reverse lexicographic order)

(Idea: Y-many copies of X



)

$$(Y, <_Y)^{(X, <_X)} := \left(\text{finite}[X,Y], <_{Y^X}\right) \text{ where:}$$

$$\text{finite}[X,Y] := \{ f \in Y^X \mid \text{the set } \{x \in X \mid f(x) \text{ is not the least element of } Y\} \text{ is finite} \}$$

$f <_{Y^X} g \iff f \neq g$ and $f(x_0) < g(x_0)$ where x_0 is the maximum element in the set $\{x \in X \mid f(x) \neq g(x)\}$

(Idea: ?)

Theorem If X, Y are well-orders then:

(1) $X+Y$ is well-ordered by $<_{X+Y}$ and

$$\operatorname{ord}(X+Y) = \operatorname{ord}(X) + \operatorname{ord}(Y)$$

(2) $X \times Y$ is well-ordered by $<_{X \times Y}$ and

$$\operatorname{ord}(X \times Y) = \operatorname{ord}(X) \cdot \operatorname{ord}(Y)$$

(3) Y^X (i.e. Finite $[X, Y]$) is well-ordered by $<_{Y^X}$ and

$$\operatorname{ord}(Y^X) = \operatorname{ord}(Y)^{\operatorname{ord}(X)}$$

the well-order exponential defined above.

Proposition (Rigidity)

(1) No well-ordered set is isomorphic to a proper initial segment of itself.

(2) Every well-ordered set has exactly one automorphism (it is rigid) : the identity.

(3) If A and B are isomorphic well-ordered sets then the isomorphism between them is unique.

Proof (1) Suppose A is isomorphic to a prop. initial seg.
By Lemma B we have an iso $f: A \to \downarrow a$.
Then $f: A \to A$ is an embedding. Hence by Lemma A $f(a) \geq a$, which contradicts $f(x) \in \downarrow a$.

(2) Suppose $f: A \to A$ is an automorphism.
Both f and f^{-1} are embeddings so $x \leq f(x)$ and $x \leq f^{-1}x$ for all x, by Lemma A. Since f is monotone $f(x) \leq f(f^{-1}(x)) = x$. So indeed $f(x) = x$ for all x.

(3) Suppose $f, g: A \to B$ are isomorphisms.
Then g^{-1} is an isomorphism from $B \to A$.
So $g^{-1} \circ f : A \to A$ is an isomorphism.
Hence by (2) $g^{-1} \circ f = id_A$.
It follows that $g = f$. ∎

We are set up to prove the classification theorem.

Proof of the classification theorem:

Every well-ordered set is isomorphic to $\downarrow\alpha$ for a unique ordinal α

Proof Let A be a well-ordered set.

Suppose $\downarrow\alpha \cong A \cong \downarrow\alpha'$.

If $\alpha' < \alpha$ then $\downarrow\alpha$ is isomorphic to a proper initial segment of itself contradicting (1) of the rigidity proposition.

Similarly $\alpha < \alpha'$ gives a similar contradiction.

So $\alpha = \alpha'$.

So if the α in the statement of the theorem exists then it is unique.

Define the relation $\underline{f} \subseteq A \times \underline{Ord}$ as follows

← in principle $\underline{f}$ is a subclass. We will soon see it is a set. So we henceforth write f.

$$(x, \alpha) \in f \iff \downarrow x \cong \downarrow\alpha$$

We establish

(i) $(x,\alpha), (x,\alpha') \in f \Rightarrow \alpha = \alpha'$ (This implies that f is a set.)

By a similar argument to at the start of the proof.

(ii) $(x,\alpha), (x',\alpha) \in f \Rightarrow x = x'$

Again by a similar argument.

By (i) and (ii) f is a bijection from $dom(f) \subseteq A$ to $image(f) \subseteq \underline{Ord}$

(iii) If $(x, \alpha) \in f$ and $x' < x$ then $x' \in dom(f)$ and $f(x') < \alpha$.

Suppose $\alpha = f(x)$ and $x' < x$

By def. of f we have an isomorphism $g : \downarrow x \to \downarrow \alpha$

Then $g\restriction_{\downarrow x'} : \downarrow x' \to \downarrow \alpha$ maps $\downarrow x'$ to an initial segment of $\downarrow \alpha$, which has the form $\downarrow \alpha'$ for some $\alpha' < \alpha$ by Lemma B.

$g\restriction_{\downarrow x'} : \downarrow x' \to \downarrow \alpha'$ is an isomorphism

So by definition of f, $f(x') = \alpha' < \alpha$.

(iv) If $(x, \alpha) \in f$ and $\alpha' < \alpha$ then $\alpha' \in image(f)$.

By a similar argument to (iii).

Given (i)–(iv), the function f is a bijection from an initial segment of A to an initial segment of $\underline{Ord}$.

And f is strictly monotone; i.e. an embedding by point (iii).

Hence f is an isomorphism from an initial segment of A to an initial segment of $\underline{Ord}$.

:

We now show that $dom(f) = A$.

Suppose not. Then $dom(f)$ is a proper initial segment of A. So $dom(f) = \downarrow x$ for some $x \in A$, by Lemma B.

Also $range(f) = \downarrow \beta$ for some $\beta \in \underline{Ord}$.

Then $f : \downarrow x \to \downarrow \beta$ is an isomorphism.

Hence $(x, \beta) \in f$ by def. of f.

So $x \in dom(f)$ contradicting that $dom(f) = \downarrow x$.

So indeed $dom(f) = A$.

$image(f) = \downarrow \alpha$ for a unique α by Lemma B.

Indeed we have that

$f : A = dom(f) \to image(f) = \downarrow \alpha$ is an isomorphism.

$\square$