Cardinals amongst ordinals

$$\omega < \omega^2 < \omega^3 < \omega^4 < \dots \quad \omega^\omega < \omega^{\omega^\omega}$$

$$< \omega^{\omega^{\omega^\omega}} < \dots \qquad \varepsilon_0 := \sup \{\underbrace{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots}_{\text{iterating } \omega \text{ times}}\}$$

$$< \varepsilon_0^{\ \omega} < \varepsilon_0^{\varepsilon_0^{\ \omega}} < \varepsilon_0^{\varepsilon_0^{\varepsilon_0^{\ \omega}}} < \dots \qquad \varepsilon_1 \qquad \text{etc.}$$

(n.b. $\varepsilon_0^{\ \omega} > \varepsilon_0$ but $\omega^{\varepsilon_0} = \varepsilon_0$)

How large do these ordinals get?

... Specifically how large in cardinality?

By the <u>cardinality of an ordinal α</u> we mean $|{\downarrow}\alpha|$

All the above ordinals are <u>countable</u> so have cardinality $\aleph_0$.

Are there any uncountable ordinals?

Theorem (Hartog's Lemma) For any set A, there exists a smallest ordinal $\underline{h}(A)$ such that $\downarrow \underline{h}(A)$ is not in bijection with any subset of A.

(Equivalently $\underline{h}(A)$ is the smallest ordinal such that there is no injective function from $\downarrow \underline{h}(A)$ to A.)

Proof Consider the set

$$W := \{ R \subseteq A \times A \mid R \text{ is a well-order on a subset of } A \}$$

Then the function on W:

$$R \longmapsto \operatorname{ord}(A', R)$$ where A' is the subset of A that R is defined on

has an image which is the set of all ordinals that are order types of well-orders on subsets of A.

Define $\underline{h}(A)$ = the smallest ordinal not in the image of the above function. $\square$

$\underline{h}(A)$ is called the <u>Hartog number</u> of A

<u>Proposition</u> For any ordinal α

$$|\underline{h}(\downarrow\alpha)| > |\downarrow\alpha|.$$

In particular $\underline{h}(\downarrow\omega)$ is an uncountable ordinal, indeed the smallest uncountable ordinal. $\omega_1 := \underline{h}(\downarrow\omega)$

Similarly $\underline{h}(\downarrow\omega_1)$ is the smallest ordinal with cardinality greater than that of $\downarrow\omega_1$. $\omega_2 := \underline{h}(\downarrow\omega_1)$.

$\omega_0 := \omega$ We get the idea to count:

$$\omega_0, \omega_1, \omega_2, \omega_3, \ldots \quad \omega_\omega, \omega_{\omega+1}, \ldots\ldots \quad \omega_\alpha$$

through the ordinals by moving to the smallest next ordinal of strictly greater cardinality.

<u>General definition of ω_α</u>

use transfinite recursion to define:

$$\omega_{(-)} : \underline{Ord} \longrightarrow \underline{Ord}$$

by

$$\omega_0 = \omega$$

$$\omega_{\alpha+1} = \underline{h}(\downarrow\omega_\alpha)$$

$$\omega_\lambda = \sup_{\alpha<\lambda} \omega_\alpha$$

<u>Idea</u> We get a notation system for ordinal representatives of cardinality classes

Def. An <u>initial ordinal</u> is an ordinal α that enjoys the property that, for every $\beta < \alpha$, $|{\downarrow}\beta| < |{\downarrow}\alpha|$.

(Equivalently, α is an initial ordinal if α is the least element of $\{\gamma \in \underline{\mathrm{Ord}} \mid |{\downarrow}\gamma| = |{\downarrow}\alpha|\}$.)

Proposition Every infinite initial ordinal is a limit ordinal.

Proof Suppose α is an infinite successor ordinal. So $\alpha = \alpha' + 1$.

But then $|{\downarrow}\alpha| = |{\downarrow}(\alpha'+1)| = |{\downarrow}\alpha'| + 1 = |{\downarrow}\alpha'|$

because α' is infinite.

This contradicts α being the smallest ordinal with its cardinality.

Theorem

1) For every α, ω_α is an initial ordinal.

2) Every initial ordinal arises as ω_α for a unique ordinal α.

Proof Exercise.

Def A set X is said to be <u>well-orderable</u> if there exists some well-order $<$ on the set X.

Suppose X is well-orderable, by $<$ say.

Then $\mathrm{ord}(X,<)$ enjoys the property that:

$$|\downarrow \mathrm{ord}(X,<)| = |X| = |\downarrow \omega_\alpha|$$

where ω_α is the initial ordinal with the same cardinality as $\mathrm{ord}(X,<)$.

We can think of ω_α as providing representatives for the cardinalities of well-ordered sets.

We introduce the convenient notation

$$\boxed{\aleph_\alpha := |\downarrow \omega_\alpha|}$$

<u>Proposition</u>

(1) For every infinite well-orderable set X, we have $|X| = \aleph_\alpha$ for a unique ordinal α.

(2) If X is s.t. $|X| = \aleph_\alpha$ then X is well-orderable

<u>Proof remarks</u> (1) As in discussion above.

(2) Given $X \cong \downarrow \omega_\alpha$, we transfer the well-ordering from $\downarrow \omega_\alpha$ to X. $\square$

Cardinals amongst the ordinals (Part 2)

In general we refer to the cardinalities of infinite well-orderable sets as <u>alephs</u>.

Theorem For every ordinal α, it holds that $\aleph_\alpha = \aleph_\alpha \cdot \aleph_\alpha$

The proof is a little bit tricky.

Before giving the proof, we show a consequence of the Theorem: the theorem completely determines the laws of cardinal arithmetic for alephs, as regards cardinal addition and multiplication.
(But not regarding cardinal expontiation.)

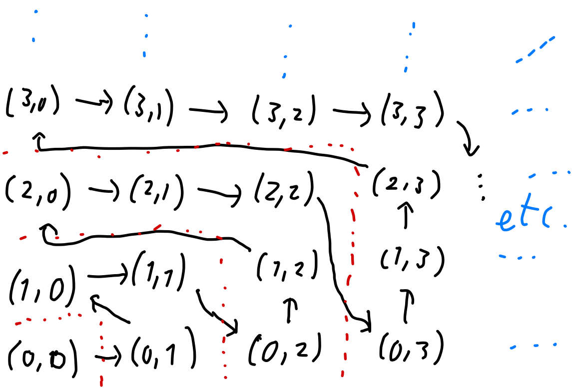
Corollary For all ordinals α, β:

(1) $\aleph_\alpha + \aleph_\beta = \aleph_{\max(\alpha,\beta)}$

(2) $\aleph_\alpha \cdot \aleph_\beta = \aleph_{\max(\alpha,\beta)}$

Proof. Exercise!

To prove the theorem we define a well-ordering on the class $\underline{Ord} \times \underline{Ord}$, which enjoys the special property that every square $\downarrow\alpha \times \downarrow\alpha \subseteq \underline{Ord} \times \underline{Ord}$ is an initial segment in the ordering. The idea behind the definition is illustrated by:



Generalising to ordinals we define

$$
\begin{aligned}
(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2) \iff \quad & \max(\alpha_1, \alpha_2) < \max(\beta_1, \beta_2) \\
\text{or } & \big(\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2) \\
& \quad \text{and } \alpha_1 < \beta_1 \big) \\
\text{or } & \big(\max(\alpha_1, \alpha_2) = \max(\beta_1, \beta_2) \\
& \quad \text{and } \alpha_1 = \beta_1 \text{ and } \alpha_2 < \beta_2 \big)
\end{aligned}
$$

Lemmas A & B below show that $\prec$ is indeed a well-ordering on $\underline{Ord} \times \underline{Ord}$ and the squares are initial segments.

Lemma A

(1) $\prec$ is a strict partial order on $\underline{Ord} \times \underline{Ord}$

(2) Every nonempty class $\underline{X} \subseteq \underline{Ord} \times \underline{Ord}$ has a minimum element under $\prec$.

Proof Exercise.

(It easily follows that $\prec$ is a total order!)

Lemma B

For any ordinal α, the set $(\downarrow\alpha) \times (\downarrow\alpha)$ is an initial segment of $\underline{Ord} \times \underline{Ord}$ under $\prec$.

Proof Exercise.

Proof of theorem Since $\aleph_\alpha \leq \aleph_\alpha \cdot \aleph_\alpha$ is obvious, we prove $\aleph_\alpha \cdot \aleph_\alpha \leq \aleph_\alpha$ by T.I. on α.

Case $\alpha = 0$ We already know that $\aleph_0 = \aleph_0 \cdot \aleph_0$.

Case $\alpha > 0$ We assume the induction hypothesis:

for all $\beta < \alpha$, $\aleph_\beta \cdot \aleph_\beta \leq \aleph_\beta$.

We need to show that $\aleph_\alpha \cdot \aleph_\alpha \leq \aleph_\alpha$.

The main step is to establish:

$$\mathrm{Ord}(\downarrow\omega_\alpha \times \downarrow\omega_\alpha, \prec) \leq \omega_\alpha \qquad (*)$$

This is well-defined by Lemmas A and B

Once we have proved $(*)$ we conclude the proof by

$$\begin{aligned}\aleph_\alpha \cdot \aleph_\alpha &= |\downarrow\omega_\alpha \times \downarrow\omega_\alpha| \\ &= |\downarrow \mathrm{Ord}(\downarrow\omega_\alpha \times \downarrow\omega_\alpha, \prec)| \leq |\downarrow\omega_\alpha| = \aleph_\alpha.\end{aligned}$$

To prove ⊛ we argue:

$$\mathrm{Ord}(\downarrow\omega_\alpha \times \downarrow\omega_\alpha, \prec)$$

$$= \sup_{\omega \leq \gamma_1, \gamma_2 < \omega_\alpha} \mathrm{Ord}(\downarrow(\gamma_1,\gamma_2), \prec)$$

because $\downarrow\omega_\alpha \times \downarrow\omega_\alpha = \bigcup_{\omega \leq \gamma_1,\gamma_2 < \omega_\alpha} \downarrow(\gamma_1,\gamma_2)$ by Lemma B

$$\leq \sup_{\omega \leq \gamma < \omega_\alpha} \mathrm{Ord}(\downarrow\gamma \times \downarrow\gamma, \prec)$$

because for any $\omega < \gamma_1, \gamma_2 < \omega_\alpha$, we have $\downarrow(\gamma_1,\gamma_2) \subseteq \downarrow\gamma \times \downarrow\gamma$ where $\gamma = \max(\gamma_1,\gamma_2)+1$ (by Lemma B and because ω_α is a limit ordinal)

But for every $\omega \leq \gamma < \omega_\alpha$ we have $|\downarrow\gamma| < \aleph_\alpha$ because ω_α is an initial ordinal.

So $|\downarrow\gamma| = \aleph_\beta$ for some $\beta < \alpha$.

By the induction hypothesis,

$\aleph_\beta \cdot \aleph_\beta \leq \aleph_\beta$ so $|\downarrow\gamma \times \downarrow\gamma| \leq |\downarrow\gamma| < \aleph_\alpha$

So $\mathrm{Ord}(\downarrow\gamma \times \downarrow\gamma, \prec) < \omega_\alpha$

Putting everything together we have:

$$\begin{aligned}
\operatorname{ord}(\downarrow\omega_\alpha \times \downarrow\omega_\alpha, \prec) &\leq \sup_{\omega \leq \gamma < \omega_\alpha} \operatorname{ord}(\downarrow\gamma \times \downarrow\gamma, \prec) \\
&\leq \omega_\alpha
\end{aligned}$$

This proves ⊛ and completes the proof.

$\square$