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DISCRETE MATHEMATICS 2

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3. Pólya theory

Now we turn our attention to a family of counting problems. These are all part of what today is known as Pólya theory of counting but they arise from very basic execercises.

Example 3.1. Assume that we have a broomstick and several cloth pieces of 5 different colours. How many different flags can be constructed using the broomstick as flagpole and two pieces of *different* colour.

Solution. We have 5 possibilities for the colour that goes next to the flagpole and 4 possibilities for the other colour, therefore we have $5 \times 4 = 20$ different flags. □

Example 3.2. If we have the same pieces of cloth as in the previous example, how many flags (with no flagpole) can be built using two pieces of cloth of *different* colour?

Solution. □

Example 3.3. We have the same piece of cloth and the broomstick, how many flags with flagpole can we build if the two colours do not need to be different.

Solution. □

Example 3.4. The same question as above but without the flagpole.

Example 3.5. We want to sit n knights in a round table. Two configurations are the same if one can rotate the table to obtain one from the other. In how many ways can the knights be sit?

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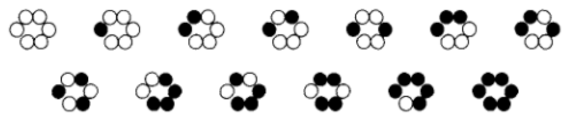

FIGURE 1. 2-colour necklaces of length 6.

Solution. □

Example 3.6. What is the number of essentially different necklaces wich can be made with with n beads of two different colours?

This is not an easy question, for $n = 6$ the number is 13. See Figure 1.

Question 3.7. What is the number of non-isomorphic graphs on n vertices?. For $n = 4$ there are 11.

Question 3.8. What is the number of essentially different ways to paint the faces (or edges, or vertices) of the cube with n colors?

We begin with with the following theorem, which was originally published by Burnside (1897) but was originally prove by Frobenius (1987).

Theorem 3.9 (Burnside's Lemma)**.** *Let G be a group acting on a set X, the number $n(G, X)$ of orbits of G on X is given by*

$$m(G, X) = \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g)$$

Proof. Count the pairs $(g, x) \in G \times X$ such that $gx = x$. A given element $g \in G$ appears in $|\mathrm{Fix}(g)|$ of those pairs. On the other hand, given $x \in X$, there are $|\mathrm{Stab}_G(x)|$ pairs with x as second coordinate. It follows that

$$\sum_{g \in G} |\mathrm{Fix}(g)| = \sum_{x \in X} |\mathrm{Stab}_G(x)|.$$

Now assume that $Y = \{y_1, \ldots, y_m\} \subseteq X$ is a set of elements, one for each orbit (so that $m(G, X) = m$).

$$\begin{aligned}
m(G, X) &= \sum_{y \in Y} 1 \\
&= \sum_{x \in Gy_1|} \frac{1}{|Gy_1|} + \cdots + \sum_{x \in Gy_m} \frac{1}{|Gy_m|} \\
&= \sum_{x \in Gy_1} \frac{1}{|Gx|} + \cdots + \sum_{x \in Gy_m} \frac{1}{|Gx|} \\
&= \sum_{x \in X} \frac{1}{|Gx|} \\
&= \sum_{x \in X} \frac{|\operatorname{Stab}_G(x)|}{|G|} \\
&= \frac{1}{|G|} \sum_{x \in X} |\operatorname{Stab}_G(x)| \\
&= \frac{1}{|G|} \sum_{g \in X} |\operatorname{Fix}(g)|.
\end{aligned}$$

□

Let us apply Burnside's Lemma to solve Example 3.5. Clearly, the cyclic group C_n acts mapping one arrangement of the table to another. Two arrangements are essentially different if and only if they are not in the same orbit. This means that we want to count orbits under the action of C_n of all the possible ways of sitting the knights. The amount of possible arrangements is $n!$. The permutation $id \in C_n$ fixes every arrangement while every non-trivial element has no fixed arrangements. By Burnside lemma the number of orbits is

$$\frac{1}{n} |\operatorname{Fix}(id)| = \frac{n!}{n}.$$

Let us now formalise this notion of conting colouring configurations of objects. Let $K = \{1, \ldots, k\}$ be a set, which we call the set of *colours*. Assume that X is a set. A *colouring* of X is a function $c : X \to K$, that is, a function that assigns the colour $c(x)$ to each element $x \in X$. The set of *colourings* of X (with colour-set K) is the set

$$K^X = \{c : X \to K\}.$$

Observe that if $|X| = n$, then $|K^X| = k^n$, as expected.

Let G be a a permutation group of X, then the group G acts on K^X by

$$\sigma c = c(\sigma^{-1}),$$

that is, for every $x \in X$, the colouring σc is given by

$$(\sigma c)(x) = c(\sigma^{-1}(x)).$$

The inverse is necessary to guarantee that the previous mapping is actually a left action.

Now we use Burnside Lemma to compute the number of orbits of colourings of a set X with respect a permutation group G of X.

Proposition 3.10. *Let X be a set with n elements and G a permutation group of S_X. Let K be a set of colours with $|K| = k$. Let m denote the number of orbits of G on K^X, the set of colourings of X. Then*

$$m = \frac{1}{|G|} \sum_{\ell} c_\ell(G) k^\ell,$$

where $c_\ell(G)$ denotes the number of elements of G that have ℓ cicles.

Proof. Let $\sigma \in G$. Assume that c is colouring fixed by σ, that is, for every $x \in X$

$$c(\sigma^{-1}x) = \sigma(c)(x) = c(x).$$

If follows that if any two elements of X in the same cycle of σ must have the same colour. Clearly, if σ has ℓ cycles, there are k^ℓ colourings of X satisfying that any two elements in the same cycle have the same colour. Any such colouring is fixed by σ. The result follows from Burnside's Lemma. $\square$

We can now solve Example 3.6. We need to compute the number $c_\ell(D_6)$ for the dihedral group D_6. The elements of D_6 are listed below:

(1)(2)(3)(4)(5)(6)	(1 2 3 4 5 6)	(1 3 5)(2 4 6)
(1 4)(2 5)(3 6)	(1 5 3)(2 6 4)	(1 6 5 4 3 2)
(1)(4)(2 6)(3 5)	(2)(5)(1 3)(4 6)	(3)(6)(2 4)(1 5)
(1 2)(3 6)(4 5)	(2 3)(1 4)(5 6)	(3 4)(2 5)(1 6)

It follows that $c_1(D_6) = 2$, $c_2(D_6) = 2$, $c_3(D_6) = 4$, $c_4(D_6) = 3$, $c_5(D_6) = 0$ and $c_6(D_6) = 1$.

The 6 permutations in the first two rows above are precisely the elements of the cycle group C_6. We can compute the numbers $c_1(C_6) = c_2(C_6) = 2$, $c_3(C_6) = 1$ and $c_6(C_6) = 1$.

We can now compute the number of necklaces of legth 6 with two colors with respect to rotations (using C_6) or with respect to rotation and flips (using D_6).

$$m(D_6) = \frac{1}{12}\left(2 \times 2 + 2 \times 2^2 + 4 \times 2^3 + 3 \times 2^4 + 1 \times 2^6\right) = \frac{156}{12} = 13$$

$$m(C_6) = \frac{1}{6}\left(2 \times 2 + 2 \times 2^2 + 1 \times 2^3 + 1 \times 2^6\right) = \frac{84}{6} = 14$$

The fact that we get one extra orbit with the cyclic group is that all but one of the orbits with respect to the dihedral group have mirror reflection (see Figure 1). The rightmost necklace in the first row does not admit mirror reflection, hence its orbit with respecct to the dihedral group has to be split into two different orbits of the cyclic group.

Proposition 3.10 allows us to count colored objects that are essentially different with respect to the symmetires of the group G. That number depends only on the cycle structure of the elements on G, more precisely on the numbers $c_\ell(G)$. In the following paragraph we will introduce the *cycle index* of a permutation group G, which generalise in some sense the numbers $c_\ell(G)$.

Let G be a permutation group of a set X, with $|X| = n$ and let $\sigma \in G$, the *cycle monomial* of σ is defined as

$$M_\sigma(t_1, \ldots, t_n) = \prod_{i=1}^{k} t_{\ell_i}$$

if σ has k cycles and the i-th cycle is of length t_{ℓ_i}. In other words, the exponent of $t_i, i \in \{1, \ldots, n\}$ in $M_\sigma(t_1, \ldots, t_n)$ is k whenever σ has k cycle of lenth i.

The *cycle index of G* is the polynomial

$$Z_G(t_1, \ldots, t_n) = \frac{1}{|G|} \sum_{\sigma \in G} M_\sigma(t_1, \ldots, t_n).$$

Observe that the cycle index actually generalises the numbers $c_\ell(G)$ discussed before. In fact, the sum of the coefficients of the terms of degree ℓ in $Z_G(t_1, \ldots, t_n)$ is precisely the number the $\frac{c_\ell(G)}{|G|}$.

From the previous discussion the following proposition is obvious.

Proposition 3.11. *Let X be a set with n elements and G a permutation group of S_X. Let K be a set of colours with $|K| = k$. Let m denote the number of orbits of G on K^X, the set of colourings of X. Then*

$$m = Z_G(k, \ldots, k).$$

We will compute some cycle index as excercises (see Exercises 3.1 to 3.4). We shall use them just to show how they work.

According to Exercise 3.2, the cycle index of the cyclic group is

$$Z_{C_n}(t_1, \ldots t_n) = \frac{1}{n} \sum_{d|n} \phi(d) t_d^{n/d}.$$

For $n = 6$ we have

$$Z_{C_6}(t_1, \ldots, t_6) = \frac{1}{6}\left(t_1^6 + t_2^3 + 2t_3^2 + 2t_6\right)$$

Similaryly for the dihedral group D_6,

$$Z_{D_6}(t_1,\dots,t_6) = \frac{1}{12}\left(t_1^6 + t_2^3 + 2t_3^2 + 2t_6 + 3t_1^2t_2^2 + 3t_2^3\right)$$

Then we can use Proposition 3.11 to solve the 2-colour necklace problem:

$$Z_{C_6}(2,\dots,2) = \frac{1}{6}\left(2^6 + 2^3 + 2\cdot 2^2 + 2\cdot 2\right) = \frac{84}{6} = 14$$
$$Z_{D_6}(2,\dots,2) = \frac{1}{12}\left(2^6 + 2^3 + 2\cdot 2^2 + 2\cdot 2 + 3\cdot 2^2\cdot 2^2 + 3\cdot 2^3\right) = \frac{156}{12} = 13$$

Proposition 3.11 allows us to compute the number of orbits of coloured objects whith no restrictions, but say that we are interested in finding 2-coloured necklaces of length 6 such that only 2 white bead are used.

From now on we will modify slightly our notation, and we shall denote $K = \{y_1,\dots,y_k\}$ the set of colours.

Definition 3.12. Let X be a set with n elements and let $G \leqslant S_n$ a permutation group acting on X. Let $K = \{y_1,\dots,y_k\}$ a set of colours. Let $\overline{v} = (n_1,\dots,n_k)$ a vector of k integers such that $n_i \geqslant 0$ for every i and $n_1 + \cdots n_k = n$. Let a_v denote the number of non-equivalent colourings (with respect to G) such that the colour y_i is used n_i times. The *pattern inventory* of G is the polynomial

$$P_G(y_1,\dots,y_k) = \sum_{\overline{v}} a_{\overline{v}} y_1^{n_1}\dots y_k^{n_k}.$$

Clearly, if we know an explicit expression for P_G then we how many colourings of a given type we have. Our task now is to find a way to compute P_G. Before doing that let us compute a very simple to show how the pattern inventory works.

Example 3.13. Let X be a set with n elements. How many colourings with 2 colours exists if we do not mind symmetry?.

Proof. Solution Let say tha the colour-set is $K = \{x, y\}$ For this problem the vectors $\overline{v}$ are of length 2 and since both entries must sum n they are of the form $(i, n-i)$. Obiously, the number of colourings using the colour x i times, when no symmetry is considered is determined by which of the elements of X are coloured with the given colour. In other words

$$a_{(i,n-i)} = \binom{n}{i},$$

hence the pattern inventoy is just

$$P_{\{id\}}(x,y) = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-1}$$

□

The expression on the right side of prevoious equations should be familiar for the reader. A well-known theorem claims that $P_{id}(x, y) = (x+y)^n$, and this is not a coincidence; let us explore this example further.

If we express the product $(x + y)^n$ as

$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$

one can see that every term of the product, before reducing similar terms, is given by chosing either x or y on each of the monomials. This explains the fact tha the coefficient of $x^i y^{n-i}$ is precisely $\binom{n}{i}$, the number of ways of chossing i times the letter x. We shall see that this coincides with the ways of chossing one colour for each cyclie of the (unique) element on the group $\{id\}$. More precisely,

Theorem 3.14 (Polya counting formula)**.** *Let X be a set of n elements and $G \leqslant S_n$ a group acting on X. Let $K = \{y_1, \cdots, y_k\}$ be a colour-set. Denote bye $P_G(y_1, \ldots, y_k)$ the pattern inventory and by $Z_G(t_1, \ldots, t_n)$ the cycle index of G, then*

$$P_G(y_1, \ldots, y_k) = Z_G\left(\sum_{j=1}^{k} y_j, \sum_{j=1}^{k} y_j^2, \ldots, \sum_{j=1}^{k} y_j^n\right).$$

Proof. Let $\overline{\mathrm{v}} = (n_1, \ldots, n_k)$ be a vector such that $n_i \geqslant 0$ for every i and $n_1 + \cdots + n_k = n$. Let $C_{\overline{\mathrm{v}}} \subseteq K^X$ the set of colourings of X such that there are exactly n_i elements of colour y_i. For $\sigma \in G$, let $C_{\overline{\mathrm{v}},\sigma} \subseteq C_{\overline{\mathrm{v}}}$ the set of colourings in $C_{\overline{\mathrm{v}}}$ that are preserved by σ.

For $\overline{\mathrm{v}}$ as above, let $\overline{\mathrm{y}}^{\overline{\mathrm{v}}}$ denote the term $y_1^{n_1} \cdots y_k^{n_k}$. If $c \in C_{\overline{\mathrm{v}},\sigma}$ then every two elements on a cycle have the same colour, because the colouring must be preserved by σ. Moreover, the lengths of the cycles coloured with a given colour y_i must add n_i. Obivously every colouring satisfying those two conditions is in $C_{\overline{\mathrm{v}},\sigma}$.

Consider the monomial $M_\sigma\left(\sum_{j=1}^k y_j, \sum_{j=1}^k y_j^2, \ldots, \sum_{j=1}^k y_j^n\right)$. This monomial is just

$$M_\sigma\left(\sum_{j=1}^{k} y_j, \sum_{j=1}^{k} y_j^2, \ldots, \sum_{j=1}^{k} y_j^n\right) = \prod_{\ell}\left(y_1^\ell + \cdots + y_k^\ell\right)$$

where ℓ runs over the lengths of the cycles of σ. Observe that the the term $\overline{\mathrm{y}}^{\overline{\mathrm{v}}}$ appears in $\prod_\ell \left(y_1^\ell + \cdots + y_k^\ell\right)$ as many times as the ways of choosing one letter (of $\{y_1, \ldots, y_k\}$) per cycle of σ such that the sum of the lengths of the cycles where we choose y_i is n_i.

When comparing the two previous analysis it is easy to see that the coefficient of $\overline{\mathrm{y}}^{\overline{\mathrm{v}}}$ in $M_\sigma\left(\sum_{j=1}^k y_j, \sum_{j=1}^k y_j^2, \ldots, \sum_{j=1}^k y_j^n\right)$ is precisely $|C_{\overline{\mathrm{v}},\sigma}|$. When we sum over all possible $\overline{\mathrm{v}}$ we have

$$M_\sigma\left(\sum_{j=1}^{k} y_j, \sum_{j=1}^{k} y_j^2, \ldots, \sum_{j=1}^{k} y_j^n\right) = \sum_{\overline{\mathrm{v}}} |C_{\overline{\mathrm{v}},\sigma}|\, \overline{\mathrm{y}}^{\overline{\mathrm{v}}}.$$

Now we sum over all possible elements $\sigma \in G$ and divide by $|G|$ and we have:

$$\begin{aligned}
Z_G\left(\sum_{j=1}^{k} y_j, \sum_{j=1}^{k} y_j^2, \ldots, \sum_{j=1}^{k} y_j^n\right) &= \frac{1}{|G|}\sum_{\sigma\in G} M_\sigma\left(\sum_{j=1}^{k} y_j, \sum_{j=1}^{k} y_j^2, \ldots, \sum_{j=1}^{k} y_j^n\right) \\
&= \frac{1}{|G|}\sum_{\sigma\in G}\sum_{\overline{\mathrm{v}}} |C_{\overline{\mathrm{v}},\sigma}|\, \overline{\mathrm{y}}^{\overline{\mathrm{v}}} \\
&= \sum_{\overline{\mathrm{v}}}\left(\frac{1}{|G|}\sum_{\sigma\in G} |C_{\overline{\mathrm{v}},\sigma}|\right) \overline{\mathrm{y}}^{\overline{\mathrm{v}}} \\
&= \sum_{\overline{\mathrm{v}}} a_{\overline{\mathrm{v}}}\overline{\mathrm{y}}^{\overline{\mathrm{v}}} \\
&= P_G(y_1, \ldots, y_k)
\end{aligned}$$

where the second to last equality follows from Burnside Lemma. □

As an example let us compute the pattern inventory of possible colouring of necklaces of length 4 using colours red (r) first with respect to the dihedral group and then with respecto to the cyclic group.

$$\begin{aligned}
Z_{D_4}(t_1, t_2, t_3, t_4) &= \frac{1}{8}\left(t_1^4 + 3t_2^2 + 2t_4 + 2t_1^2 t_2\right) \\
Z_{C_4}(t_1, t_2, t_3, t_4) &= \frac{1}{4}\left(t_1^4 + t_2^2 + 2t_4\right)
\end{aligned}$$

Using Polya counting formula we can see that

$$\begin{aligned}
P_{D_4}(r, g, b) &= Z_{D_4}(r + g + b, r^2 + g^2 + b^2, r^3 + g^3 + b^3, r^4 + g^4 + b^4) \\
&= \frac{1}{8}\big((r + g + b)^4 + 3(r^2 + g^2 + b^2)^2 + 2(r^4 + g^4 + b^4) + \\
&\qquad 2(r + g + b)^2(r^2 + g^2 + b^2)\big) \\
&= r^4 + r^3 g + 2r^2 g^2 + rg^3 + g^4 + r^3 b + 2r^2 gb + 2rg^2 b + g^3 b + \\
&\qquad + 2r^2 b^2 + 2rgb^2 + 2g^2 b^2 + rb^3 + gb^3 + b^4
\end{aligned}$$

and

$$
\begin{aligned}
P_{C_4}(r,g,b) &= Z_{C_4}(r+g+b, r^2+g^2+b^2, r^3+g^3+b^3, r^4+g^4+b^4) \\
&= \frac{1}{4}\left((r+g+b)^4 + (r^2+g^2+b^2)^2 + 2(r^4+g^4+b^4)\right) \\
&= r^4 + r^3g + 2r^2g^2 + rg^3 + g^4 + r^3b + 3r^2gb + 3rg^2b + g^3b + \\
&\qquad\qquad + 2r^2b^2 + 3rgb^2 + 2g^2b^2 + rb^3 + gb^3 + b^4
\end{aligned}
$$

Observe that there are two non-equivalent colouring with two red beads, one green and one blue under D_4 wheareas there are three of them under C_4. This is essentially because the necklaces (r,r,g,b) and (r,r,b,g) are different with respecto to the cyclic group but equivalent with resepct to the dihedral group.

Consider now the following example:

Example 3.15. Suppose a jewelry company plans to market a new line of unisex bracelets. The bracelets are sold in pairs, for a couple to share. Each bracelet consists of n beads, some gold and some silver, and the two bracelets in a pair are opposites, in the sense that one can be obtained from the other by changing each silver bead to a gold one and each gold to a silver. For example, if one bracelet has two adjacent gold beads and $n-2$ silver beads, then its mate has two adjacent silver beads and $n-2$ gold beads.

Using the cycle index of D_4 computed before we can see that there are 6 non equivalent bracelets with two colours. Moreover, using Polya counting formula we can easily see that the pattern inventory is

$$
P_{D_4}(g,s) = s^4 + s^3g + 2s^2g^2 + sg^3 + g^4
$$

Meaning that bracelets can be represented as (s,s,s,s), (s,s,s,g), (s,g,s,g) (s,s,g,g), (s,g,g,g) and (g,g,g,g). However, when we consider the change of colours we only have three different pairs, namelly

$$
\begin{gathered}
\{(s,s,s,s),(g,g,g,g)\} \\
\{(s,g,s,g),(g,s,g,s)\} \\
\{(s,s,g,g),(g,g,s,s)\} \\
\{(g,g,g,s),(s,s,s,g)\}
\end{gathered}
$$

Notice that the pair of the bracelet (s,g,s,g) is (g,s,g,s), which is equivalent to the former under the action of D_4. In other words, that pair of bracelets consists of two identical bracelets.

We can take a step back and count unfasten bracelets (that is, linear $\{g,s\}$-sequences) and then consider two of them equivalet if we can obtain one from the other by either the action of a dihedral group or by a swich of colours. We can see that we obtain precisely the same equivalence classes as before:

$\{(s,s,s,s),(g,g,g,g)\}$
$\{(s,g,s,g),(g,s,g,s)\}$
$\{(s,s,g,g),(g,g,s,s),(s,g,g,s),(g,s,s,g)\}$
$\{(g,g,g,s),(g,g,s,g),(g,s,g,g),(s,g,g,g),(s,s,s,g),(s,s,g,s),(s,g,s,s),(g,s,s,s)\}$

Observe that in these consists of the orbits of linear $\{g,s\}$-sequences under the action of two groups. On the one hand we have the group D_4 acting as symmetries of the necklaces and on the other hand we have a cyclic group C_2 swapping the colours. Let us formalise these ideas for the general case.

Let X be a set with n elements and $G \leqslant S_n$ a permutation group acting on X. Let K be a set with k *colours* and $H \leqslant S_k$ a permutation group acting on K. If $\sigma \in G, \tau \in H$ and $c \in K^X$ the mapping $(\sigma,\tau)c \mapsto \bar{c}$ where $\bar{c}$ is the colouring defined by

$$\bar{c}(x) = \tau\left(c(\sigma^{-1}x)\right)$$

defines a left action of $G \times H$ on K^X (see Exercise 3.14).

The following result tells us how to count the number of orbits for this action.

Proposition 3.16. *Let X be a set with $|X| = n$ and $G \leqslant S_n$ a permutation group acting on X. Let K be a set of colours with $|K| = k$ and $H \leqslant S_k$ a permutation group acting on K. The number of orbits N of K^X under the action on $G \times H$ defined above is:*

$$N = \frac{1}{|H|}\sum_{\tau\in H} Z_G(m_1(\tau),\dots,m_n(\tau))$$

where $m_i(\tau) = \sum_{j|i} j \cdot z_j(\tau)$ and $z_j(\tau)$ denotes the number of cycles of length j in τ.

Proof. Let $\sigma \in G$ and $\tau \in H$. Denote by $\phi(\sigma,\tau)$ the number of colourings preserved by (σ,τ). By Burnside lemma

$$N = \frac{1}{|G\times H|}\sum_{(\sigma,\tau)\in G\times H}\phi(\sigma,\tau) = \frac{1}{|H|}\sum_{\tau\in H}\left(\frac{1}{|G|}\sum_{\sigma\in G}\phi(\sigma,\tau)\right)$$

It remains to show that

$$\frac{1}{|G|}\sum_{\sigma\in G}\phi(\sigma,\tau) = Z_G(m_1(\tau),\dots,m_n(\tau)).$$

Let $\gamma_1,\dots,\gamma_d$ be the cycle of σ and let us denote $V_i = supp(\gamma_i) \subseteq X$ the *support* of γ_i, that is $V_i = X \setminus \mathrm{Fix}(\gamma_i)$. Clearly a colouring c is preserved by (σ,τ) if and only if each of its restriction $c_i : V_i \to K$ is preserved by (γ_i,τ). Let $\phi_i(\sigma,\tau)$ be the number of colourings of V_i preserved by (γ_i,τ). From the previous observation we have that $\phi(\sigma, \tau) = \prod_{i=1}^{d} \phi_i(\gamma_i, \tau)$. Observe that if $c_i : V_i \to X$ is a colouring fixed by (γ_i, τ) then

$$c_i(x) = (\gamma_i, \tau)c_i(x) = \tau(c_i(\gamma_i^{-1}x)) = \tau(c_i(\sigma^{-1}x)) \text{ for all } x \in V_i,$$

or equivalently

$$c_i(\sigma y) = \tau(c_i(y)) = \text{ for all } y \in V_i$$

Assume that γ_i has length ℓ_i (that is $|V_i| = \ell_i$), pick $x_0 \in V_i$ and let $k := c_i(x_0)$. Observe that the colour of every other element in V_i depends only on k and τ. In fact, if $y = \sigma^r(x_0)$ then $c_i(y) = c_i(\sigma^r x_0) = \tau^r(c_i(x_0)) = \tau^r(k)$. In particular, since this is true for $r = \ell_i$, we have that the cycle of τ containing k must divide ℓ_i.

Conversely, if we pick any element $k \in K$ such that the cycle of τ containing k has length a divisor of ℓ_i, then the mapping $c_i : V_i \to X$ defined by $c_i(\gamma_i^r x_0) = \tau^r(k)$ is a well-defined colouring that is preserved by (γ_i, τ).

In other words the number $\phi_i(\sigma, \tau)$ is equal to the number of ways of picking a colour k lying on a cycle of τ of length a divisor of ℓ_i. Obviously this number is

$$\phi_i(\sigma, \tau) = \sum_{j|\ell_i} j z_j(\tau) = m_{\ell_i}(\tau)$$

Finally, observe that

$$\phi(\sigma, \tau) = \prod_{i}^{d} \phi_i(\sigma, \tau) = \prod_{i}^{d} m_{\ell_i}(\tau) = M_\sigma(m_1(\tau), \ldots, m_n(\tau)).$$

The result follows from taking the sum over the elements of G and dividing by $|G|$ ☐

We finish the section of the notes by remarking that the previous result solves the problem of finding the number of non-equivalent colourings under the action of the group G on the set of elements and the group H on the set of colours. The notion of a *patter inventory* can be also extended to this context and an analogous result to Theorem 3.14 was proved by De Brujin in 1964. We leave this result out of these notes but is the reader is interested a proof can be found on [Bru67]

Exercises.

3.1 Find the cycle index of the group of rotations of the cube.

3.2 Prove that the cycle index if the cyclic group is

$$Z_{C_n}(t_1, \ldots t_n) = \frac{1}{n} \sum_{d|n} \phi(d) t_d^{n/d}.$$

3.3 Prove that the cycle index of the dihedral group $D_n$is

$$Z_{D_n}(t_1,\dots,t_n)=\begin{cases}\frac{1}{2n}\left(\sum_{d|n}\phi(d)t_d^{n/d}+nt_1t_2^{\frac{n-1}{2}}\right) & \text{for } n \text{ odd,}\\ \frac{1}{2n}\left(\sum_{d|n}\phi(d)t_d^{n/d}+\frac{n}{2}t_1^2t_2^{\frac{n}{2}-1}+\frac{n}{2}t_2^{\frac{n}{2}}\right) & \text{for } n \text{ even}\end{cases}$$

3.4 We say that a partition P of $[n]$ is of type $(k_1, k_2, \dots k_n)$ if P has k_1 subsets of size 1, k_2 of size 2, etc.

(a) Find the number of partitions of $[n]$ of a given type $(k_1, \dots k_n)$.

(b) Use the previous item to prove that

$$Z_{S_n}(t_1,\dots,t_n)=\sum_{(k_1,\dots k_n)}\frac{1}{1^{k_1}2^{k_2}\cdots n^{k_n}k_1!k_2!\cdots k_n!}t_1^{k_1}\cdots t_n^{k_n}.$$

(c) Compute explicitly the cycle index of S_n for $n \leqslant 5$.

3.5 Find the number of essentially different colourings of the vertices of K_5 (the complete graph with 5 vertices) with at most 5 colours.

3.6 Let X_1 and X_2 be two disjoint sets of size n_1 and n_2, respectivelly. Let G_i a permutation group of the set X_i ($i \in \{1, 2\}$). Consider the group $G = G_1 \times G_2$.

(a) Prove that G acts faithfully on $X = X_1 \cup X_2$.

(b) Prove that

$$Z_G(t_1,\dots,t_{n_1},s_1,\dots,s_{n_1})=Z_{G_1}(t_1,\dots,t_{n_1})\cdot Z_{G_2}(s_1,\dots,s_{n_2})$$

(c) Compute the cycle index of the largest subgroup of S_5 that preserves the partition $\{\{1, 2, 3\}, \{4, 5\}\}$.

3.7 The commander of a space cruiser wishes to post four sentry ships arrayed around the cruiser at the vertices of a tetrahedron for defensive purposes, since an attack can come from any direction.

(a) How many ways are there to deploy the ships if there are two different kinds of sentry ships available, and we discount all symmetries of the tetrahedral formation?

(b) How many ways are there if there are three different kinds of sentry ships available?

3.8 Consider the natural action of the dihedral group D_4 on the tiles of an unpainted 4×4 chekerboard. Let G denote the induced permutation group of the 16 tiles.

(a) Find the cycle index of G.

(b) In how many ways can we paint the board if we use colours black and white.

(c) In how many ways can we paint the checkerboard if 8 tiles must be black and 8 must be white?

(d) What if we paint exactly 4 tiles black and there must be exactly one black tile on each row and each column?

3.9 Consider the symmetric group S_4 acting on the 6 edges of a complete graph K_4. Let G denote the induced permutation group, that is $G \leqslant S_6$.
 (a) Compute the cycle index of G.
 (b) Use this to compute the number of non-isomorphic graphs on 4 vertices.

3.10 How many 0, 1-sequences of lenght 12 exists if two sequences are considered to be the same if one can be obtained from the other by a cyclic shift. How many are there if each consists of exactly 6 ones and 6 zeros.

3.11 What is the pattern inventory for coloring n objects using the m colours $y_1, \dots, y_m$ if the group of symmetries is S_n?

3.12 Two identical cubes are glued to two oposite faces of a third cube to form a 3×1 prism. The prism has 14 squares exposed (4 of the cube in the middle 5 of each of the other two cubes).
 (a) Find the permutation group G on the 14 squares induced by all the possible ways of rotating the prism.
 (b) Compute the cycle index of G.
 (c) In how many ways can the squares be painted using at most three colours: black, white and blue.
 (d) Whay if exactly two of the squares must be blue?

3.13 What is the number of essentially different ways to paint the faces of a cube such that one face is red, two are blue, and the remaining three are green?

3.14 Let X be a set with n elements and $G \leqslant S_n$ a permutation group acting on X. Let K be a set with *k colours* and $H \leqslant S_k$ a permutation group acting on K. Prove that if $\sigma \in G$, $\tau \in H$ and $c \in K^X$ the mapping $(\sigma, \tau)c \mapsto \bar{c}$ where $\bar{c}$ is the colouring defined by

$$\bar{c}(x) = \tau\left(c(\sigma^{-1}x)\right)$$

defines a left action of $G \times H$ on K^X.

3.15 The hydrocarbon naphthalene has ten carbon atoms arranged in a double hexagon as in Figure 2, and eight hydrogen atoms attached at each of the positions labeled 1 through 8.

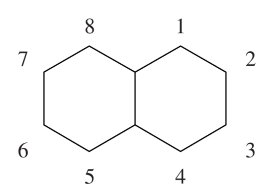

FIGURE 2. Naphthalene

(a) Naphthol is obtained by replacing one of the hydrogen atoms of naphthalene with a hydroxyl group (OH). How many isomers of naphtholare there?
(b) Tetramethylnaphthalene is obtained by replacing four of the hydrogen atoms of naphthalene with methyl groups (CH_3). How many isomers of tetramethylnaphthalene are there?
(c) How many isomers may be constructed by replacing three of the hydrogen molecules of naphthalene with hydroxyl groups, and another three with methyl groups?
(d) How many isomers may be constructed by replacing two of the hydrogen molecules of naphthalene with hydroxyl groups, two with methyl groups, and two with carboxyl groups ($COOH$)?

3.16 Determine the number of ways to color the faces of a cube using the three colors red, blue, and green, if two colorings are considered to be equivalent if one can be obtained from the other by rotating the cube in some way in three-dimensional space, and possibly exchanging green and red.

References.

[Bru10] Richard Brualdi. *Introductory combinatorics*. Upper Saddle River, N.J: Pearson/Prentice Hall, 2010. ISBN: 0136020402.

[Bru67] N. G. de Bruijn. “Color patterns that are invariant under a given permutation of the colors”. en. In: *Journal of Combinatorial Theory* 2.4 (June 1967), pp. 418–421. ISSN: 0021-9800. DOI: 10.1016/S0021-9800(67)80052-8. URL: https://www.sciencedirect.com/science/article/pii/S0021980067800528 (visited on 04/10/2022).

[HHM10] John Harris, Jeffry L. Hirst, and Michael Mossinghoff. *Combinatorics and Graph Theory*. SPRINGER NATURE, Dec. 2010. 400 pp. ISBN: 1441927239. URL: https://www.ebook.de/de/product/13975405/john_harris_jeffry_l_hirst_michael_mossinghoff_combinatorics_and_graph_theory.html.

[J H09] R. M. Wilson J. H. Van Lint. *A Course in Combinatorics*. Cambridge University Press, Mar. 2009. 620 pp. ISBN: 0521006015. URL: https://www.ebook.de/de/product/3258175/j_h_van_lint_r_m_wilson_a_course_in_combinatorics.html.

[Mer03] Russell Merris. *Combinatorics*. John Wiley & Sons, Inc., Aug. 2003. DOI: 10.1002/0471449687.