Matematika – 3. kolokvij

(1)

1 a) $\lim\limits_{x\to 0} \frac{x+\sin x}{x} = c$

$\lim\limits_{x\to 0}\left(1+\frac{\sin x}{x}\right) = 1+1 = \underline{\underline{2}} \Rightarrow \boxed{c=2}$

5

b) Težava z odvedljivostjo je lahko sila v 0. Poglejmo po definiciji

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\frac{h+\sin h}{h}-2}{h}$$

$$= \lim_{h\to 0}\frac{\sin h - h}{h^2} \overset{L'H}{=} \lim_{h\to 0}\frac{\cos h - 1}{2h} \overset{L'H}{=} \lim_{h\to 0}\frac{-\sin h}{2}$$

$= 0 \Rightarrow$ limita obstaja, torej je f odvedljiva v 0

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velja

$$f'(x) = \begin{cases} \frac{\cos x \cdot x - \sin x}{x^2} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

c) zvezna odvedljivost ②

ali je $\lim\limits_{x\to 0} \frac{\cos x \cdot x - \sin x}{x^2} = 0$?

$\lim\limits_{x\to 0} \frac{x\cos x - \sin x}{x^2} \overset{L'H}{=} \lim\limits_{x\to 0} \frac{\cos x - x\sin x - \cos x}{2x}$

$= \lim\limits_{x\to 0} \frac{-x\sin x}{2x} = \lim\limits_{x\to 0} -\frac{\sin x}{2} = 0$ ✓

$\Rightarrow$ f je zvezno odvedljiva

(S) (S)

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② Naloga

$y^2 e^{2x} = ay + x^2 + 6$; točka $T(a,3)$

leži na krivulji

$9 = a \cdot 3 + 6 \quad (*)$ (S)

odvod:

$2yy' e^{2x} + y^2 e^{2x} \cdot 2 = ay' + 2x$

$y'(2ye^{2x} - a) = 2x - 2e^{2x}y^2$ (S)

$\Rightarrow y' = \frac{2x - 2e^{2x}y^2}{2ye^{2x} - a} \overset{x=0,\ y=3}{=} \frac{-18}{6-a}$

želimo, da je tangenta $\perp$ na premico

$$\underbrace{x-6y-1=0}$$

$$6y=x-1$$

$$y=\frac{1}{6}x-\frac{1}{6} \Rightarrow k=\frac{1}{6}$$

$$k_t=\frac{-18}{6-a}=-\frac{1}{k}=-6 \quad 5$$

$$\Rightarrow -18=-36+6a$$

$$18=6a \Rightarrow \underline{\underline{a=3}} \quad \text{vstavo v } (x)$$

$$b=9-3a=9-9=0 \Rightarrow \underline{\underline{b=0}} \quad 5$$

3.) a) $\lim\limits_{x\to 1} \ln x \ln(x-1) = \text{"}0\cdot(-\infty)\text{"}$

$$=\lim_{x\to 1}\frac{\ln(x-1)}{\frac{1}{\ln x}} \overset{L'H}{=} \lim_{x\to 1}\frac{\frac{1}{x-1}}{\frac{-1}{\ln^2 x}\cdot\frac{1}{x}}$$

$$=\lim_{x\to 1}\frac{-\cancel{x}^{\to 1}\ln^2 x}{x-1}=\lim_{x\to 1}\frac{\ln^2 x}{1-x}\overset{L'H}{=}$$

5

$$\lim_{x\to 1}\frac{2\ln x\cdot\frac{1}{x}}{-1}=\lim_{x\to 1}-2\frac{\ln x}{x}=\underline{\underline{0}} \quad +5$$

5) $\lim\limits_{x\to 0} (\cos x)^{\frac{1}{x^2}} = \lim\limits_{x\to 0} e^{\frac{1}{x^2}\ln(\cos x)}$ (4)

$= e^{\lim\limits_{x\to 0}\frac{1}{x^2}\ln(\cos x)}$

Izračunajmo $\lim\limits_{x\to 0} \frac{1}{x^2}\ln(\cos x) \overset{L'H}{=}$

$$\lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot\overbrace{\left(-\frac{\sin x}{x}\right)}^{\to 1}}{2} = \frac{1\cdot(-1)}{2} = -\frac{1}{2}$$

$$\Rightarrow \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

(10)

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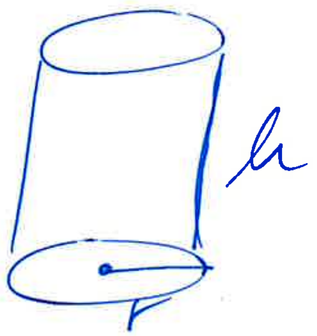
(4)



$V = \pi r^2 h$

minimiziramo površinu

$2\pi r^2 + 2\pi r\cdot h$

$= 2\pi r^2 + 2\cancel{\pi}\cancel{r}\cdot\frac{V}{\cancel{\pi} r^{\cancel{2}}}$

$\Rightarrow f(x) = 2\pi x^2 + \frac{2V}{x}$ (10)

$f'(x) = 4\pi x - \frac{2V}{x^2} = 0$

$= \frac{4\pi x^3 - 2V}{x^2} = 0 \quad \Rightarrow \quad 2\pi x^3 = V$

$x = \sqrt[3]{\frac{V}{2\pi}}$ (5)

5

$$\Rightarrow r = \sqrt[3]{\frac{V}{2\pi}} = \sqrt[3]{\frac{\pi r^2 h}{2\pi}} = \sqrt[3]{\frac{r^2 h}{2}}$$

$V = \pi r^2 h$

$$\Rightarrow r^3 = \frac{r^2 h}{2} \Rightarrow \boxed{h = 2r}$$ 5

$\Rightarrow$ razmerje med premerom in višino valja
mora biti $\underline{\underline{1:1}}$

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5 $f(x) = \arctan\left(\frac{x}{x-1}\right)$

$D_f = \mathbb{R} \setminus \{1\}$ 2

sodost/lihost: arctan je liha,

$$g(x) = \frac{x}{x-1}, \quad g(-x) = \frac{-x}{-x-1} = \frac{x}{1+x}$$

ni soda, ne liha

$\Rightarrow$ isto velja za f

ničle: $\frac{x}{x-1} = 0 \Rightarrow \underline{x = 0}$ 2

poli: ? preverimo ugpej limito

$$\lim_{x \downarrow 1} \arctan\left(\frac{x}{x-1}\right) = \arctan\left(\lim_{x \downarrow 1} \frac{x}{x-1}\right)$$

$$= \arctan(\infty) = \frac{\pi}{2}$$ 2

$$\lim_{x \uparrow 1} \arctan\left(\frac{x}{x-1}\right) = \arctan\left(\lim_{x\uparrow 1} \frac{x}{x-1}\right)$$ 6

$$= \arctan(-\infty) = -\frac{\pi}{2}$$ 4

$\Rightarrow$ f nima pola 2

$$\lim_{x\to\infty} f(x) = \arctan\left(\lim_{x\to\infty} \frac{x}{x-1}\right) = \arctan 1 = \frac{\pi}{4}$$

$$\lim_{x\to-\infty} f(x) = \arctan\left(\lim_{x\to-\infty} \frac{x}{x-1}\right) = \arctan\left(\lim_{x\to\infty} \frac{-x}{-x-1}\right)$$

$$= \arctan(+1) = +\frac{\pi}{4}$$ 2

$\Rightarrow$ desna vodoravna asimptota je $y = \frac{\pi}{4}$,
leva vodoravna asimptota je $y = +\frac{\pi}{4}$

ekstremi:

$$f'(x) = \frac{1}{1+\left(\frac{x}{x-1}\right)^2} \cdot \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2+x^2} < 0 \quad \forall x$$

$\Rightarrow$ ni ekstremov in funkcija je povsod padajoča 2 prevoji: $f''(x) = \dfrac{1}{((x-1)^2+x^2)^2} \cdot (2(x-1)+2x)$

$f''(x) = 0 \Rightarrow 2(x-1)+2x = 0$

2

$2x - 2 + 2x = 0$

$4x = 2$

$x = \frac{1}{2}$ je prevoj

(v $\frac{1}{2}$ gre funkcija iz konkavnosti v konveksnost)

konveksnost:

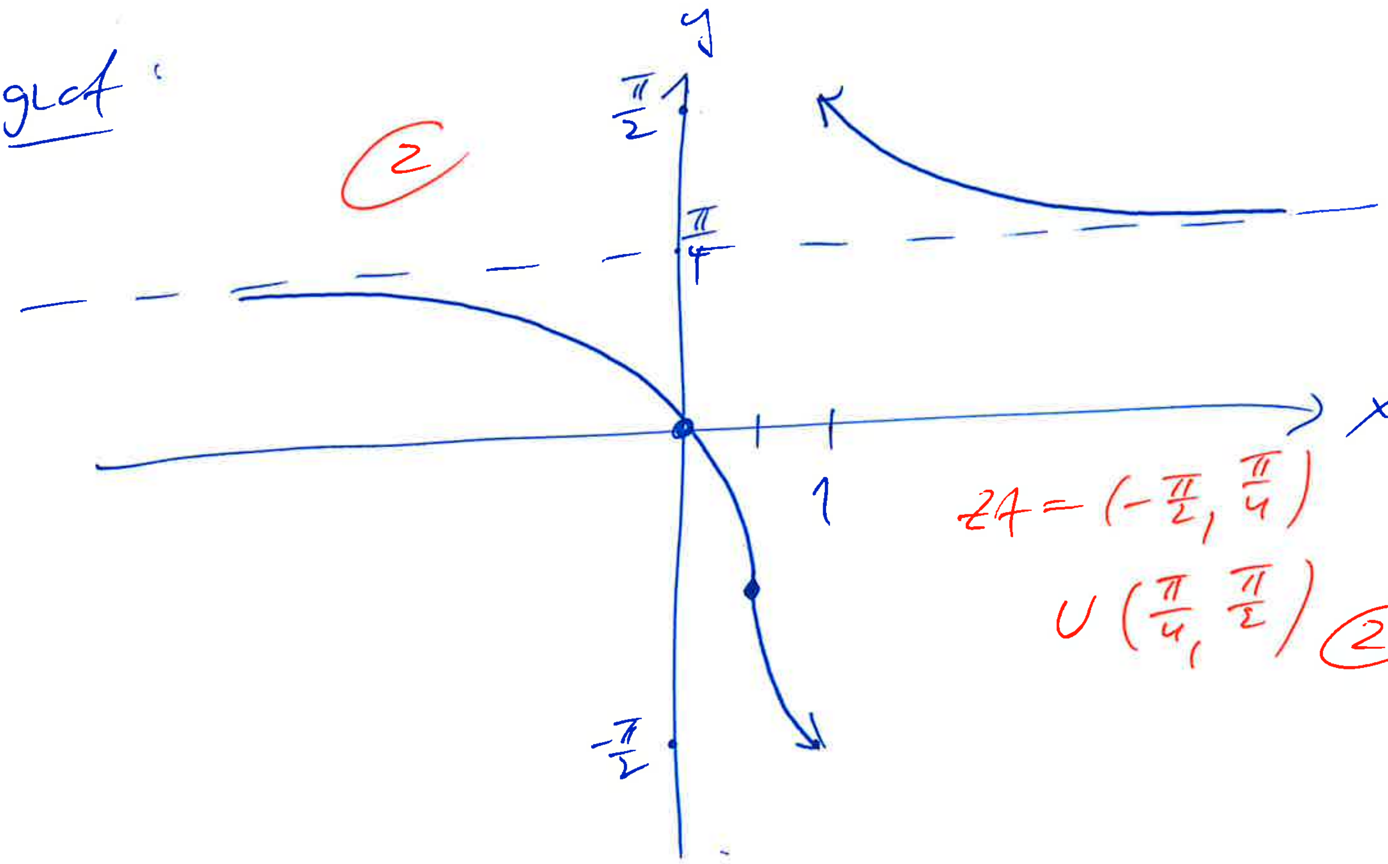
$f''(x) > 0 \Rightarrow \dfrac{4x-2}{((x-1)^2+x^2)^2} > 0$ za $x > \frac{1}{2}$

2

konkavna za $x < \frac{1}{2}$

graf:

2



$Z_f = (-\frac{\pi}{2}, \frac{\pi}{4}) \cup (\frac{\pi}{4}, \frac{\pi}{2})$

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