University *of Ljubljana*
Faculty *of Mathematics and Physics*

Discrete Mathematics 2
Prvi kolokvij

Name: ______________________________

Student Number: ______________________________

Question:	1	2	3	4	5	6	Total
Points:	20	20	20	20	20	20	120
Score:							

Instructions

1. Read this instructions before you start. **DO NOT TURN THE PAGE UNTIL INDICATED**
2. Please write your name and student number in the spaces above.
3. You must remain seated during the entire exam and you are not allowed to communicate with your classmates.
4. You are allowed to use the class notes provided by the instructor but no other source of information is allowed.
5. Please, mute your phone and try to use it only in case of an emergency.
6. The exam consists of 6 problems. You only need to solve 5 but you are allowed (and encouraged) to try (and solve) all of them. Any extra job will count, meaning that you can get up to 20% of extra marks.
7. Please use blank pages to write your solutions. Write your name on each of them. Yo can ask the instructor for more paper sheets if you need.
8. Try no to solve more than one problem on each side of the page, but you can use each side for a different problem. Please indicate very clearly which problem you are solving on each page, use sentences like 'Problem 3, page 1/2'.
9. *Show your work*: write answers as complete as possible. This improves your chances to get partial marks. You do not need to compute numerical operations, meaning that an aswer as $3\times 6^{17}+5^3+2+1$ is perfectly valid as long as you justify where it came from. If you just give a large number such as 352039485 without explaining where it came from, you will not get full mark (even if it is correct!)
10. **Good luck!**

1. (20 Pts.) Prove that if two permutations are conjugate then they have the same number of fixed points. Give an example of two permutations with the same number of fixed points that are not conjugate.

Solution:

   1. *If σ and τ are permutations of the same* finite *set then they have the same cycle type. Obviously they have the same number of fixed points.*

   2. *If $\sigma = \mu\tau\mu^{-1}$ and x is a fixed point of τ then $\mu(x)$ is a fixed point of σ, indeed:*

$$\begin{aligned}\sigma(\mu(x)) &= \mu\tau\mu^{-1}(\mu(x)) \\ &= \mu\tau(x) \\ &= \mu(x).\end{aligned}$$

In S_5 the permutations $(1\ 2\ 3\ 4)$ and $(1\ 2)(3\ 4)$ have exactly one fixed point (5) and theay are not conjugate.

2. (20 Pts.) Let G be a group acting transitively on a set X. Let H be a subgroup of G and let S denote the subgroup $\mathrm{Stab}_G(x)$. Prove that the following statements are equivalent

   (i) $G = SH$,

   (ii) $G = HS$,

   (iii) H is transitive.

   Use the above to prove that the only transitive subgroup containing S is the group G itself.

Solution:

(i) $\Rightarrow$ (ii) *Let $g \in G$, then $g^{-1} \in G$, which implies that there exists $s \in S$ and $h \in H$ such that $g^{-1} = sh$. Then*

$$g = \left(g^{-1}\right)^{-1} = (sh)^{-1} = h^{-1}s^{-1} \in HS.$$

It follows that $G = HS$.

(ii) $\Rightarrow$ (iii) *Let $y \in X$, we will show that for every $y \in Y$ there exists $h \in H$ such $hx = y$. This is equivalent to show that H is transitive. Since G is transitive, there exists $g \in G$ such that $gx = y$; since $G = HS$, then $g = hs$ for some $h \in H$ and $s \in S$, finally:*

$$hx = h(sx) = (hs)x = gx = y.$$

(iii) $\Rightarrow$ (i) *Let $g \in G$ and let $y = g^{-1}x$, hence $x = gy$. Since H is transitive, there exists $h \in H$ such that $x = hy$ (implying $y = h^{-1}x$). Obseve that*

$$(gh^{-1})x = g(y) = x.$$

That means that $gh^{-1} \in S$, hence $g = sh$ for some $s \in S$. It follows that $G = SH$.

If $H \leqslant G$ and contains S then $HS = H$ but if H is transitive, then $G = HS = H$.

3. (20 Pts.) Let G be a group acting transitively on a set X. Let $x \in X$ and denote by S the subgroup $\mathrm{Stab}_G(x)$. Prove that if G acts transitively then

$$[N_G(S) : S] = |\mathrm{Fix}(S)|$$

Here $N_G(S)$ denotes the *normaliser* of S, that is, the largest subgroup of G in which S is normal and $\mathrm{Fix}(S) = \{y \in X : gy = y \ \forall g \in S\}$.

Solution: *Consider the function $\phi : N_G(S)/S to Fix(S)$ given by $\phi(nS) = nx$.*

This function is well-defined: First observe that if $n \in N_G(S)$ then $nx \in \mathrm{Fix}(S)$. Take $s \in S$, now observe that since $n \in N_G(S)$, then there exists $s' \in S$ such that $n^{-1}sn = s'$, equivalently, $sn = ns'$ then $(sn)x = (ns')x = nx$. Then $nx \in \mathrm{Fix}(S)$.

Take two elements $n_1, n_2 \in N_G(S)$ then,

$$n_1S = n_2S \Leftrightarrow n_1^{-1}n_2 \in S \Leftrightarrow n_1x = n_2x$$

Reading the arrows of the equation above shows that the definition of ϕ does not depend on the representant of the coset. If we read the arrows in the opposite direction we show that ϕ is injective.

Take $n \in G$ such that $nx \in \mathrm{Fix}(S)$. We have that $snx = nx$ for every $s \in S$. Therefore $n^{-1}sn \in S$. It follows that $n \in N_G(S)$. Since G acts transitively, then every element in $\mathrm{Fix}(S)$ is of the form nx for some $n \in G$. This proves that ϕ is surjective.

4. (20 Pts.) How many essentially different ways are there to label the faces of a cube with the numbers 1 through 6 ...

   (i) ... if each number may be used more than once, but not all numbers need to be used?

   (ii) ... if each number may only be used once?

   Note: two labellings of the cube are considered to be the same if we can rotate the cube on the space and obtain one where the values on each face are the same as in the other, regardless of the particular orientation of each of the numbers.

Solution: *Let $X = \{A, B, C, D, E, F\}$ be the faces of the cube and $K = \{1, 2, 3, 4, 5, 6\}$ the set of possible numbers. The group of rotations of the cube acts transitivelly on faces, edges and vertices. Moreover, if u, v are vertices (resp. edges, faces) then every element of this group maps $\{u, v\}$ to a pair of opposite vertices (resp. edges, labels). It follows that if H_1 and H_2 are the cyclic groups of rotations with axis the line segment through (the centres of) a pair of vertices (resp. edges, faces), then H_1 and H_2 are conjugate in G. It follows that the group G is a permutation group on X with elements having the following cycle-type:*

(a) The identity, of type $[1, 1, 1, 1, 1, 1]$.

(b) 8 rotations of period 3 with axis through a pair of opposite vertices. Those are or type $[3, 3]$

(c) 6 half turns with axis through the midpoints of opposite edges of type $[2, 2, 2]$.

(d) Two rotations of period 4 (one clockwise and its inverse) fixing a pair of opposite faces; this gives us 6 elements of type $[4, 1, 1]$.

(e) *One rotation of period 2 for each pair of opposite faces, for a total of 3 elements of type* $[2, 2, 1, 1]$.

We have a total of 24 *elements.*

From the observations above we can easily compute the cycle index of G*:*

$$Z_G(t_1, \dots, t_6) = \frac{1}{24}(t_1^6 + 8t_3^2 + 6t_2^3 + 6t_1^2 t_4 + 3t_1^2 t_2^2)$$

(i) *This is the same as counting the orbits of* G *on* K^X. *We know that this number is*

$$Z_G(6, \dots, 6) = \frac{53424}{24} = 2226.$$

(ii) *We use Burnside Lemma to solve this part. We need to count the number* N *of orbits of the action of* G *on the set*

$$Y = \{f : \{A, B, C, D, E, F\} \to \{1, 2, 3, 4, 5, 6\} \mid f \text{ is injective}\}.$$

By Burnside Lemma

$$N = \frac{1}{|G|} \sum_{\sigma \in G} |Fix(\sigma)|.$$

Given $\sigma \in G$ *and* $f \in Y$, $f \in \text{Fix}(\sigma)$ *if and only if* $f(\sigma^{-1}(x)) = f(x)$ *for every* $x \in X$. *Since* f *is injective, it means that* $\sigma^{-1}(x) = x$ *for every* x *or equivalently* $\sigma(x) = x$. *This implies that* $\sigma = id$ *and hence*

$$N = \frac{\text{Fix}(id)}{|G|} = \frac{6!}{24} = 30$$

5. (20 Pts.) In how many ways can we put together a necklace of six corals if we have corals available in white, red and blue? What if we want to use two reds and four blue corals? What if we want to use three white, one red and two blue corals? Two necklaces are considered the same if we can get one from the other either by rotations or by a flip.

Solution: *We want to count the orbits of necklaces under the dihedral group* D_6. *We know the cycle type of this group:*

$$Z_{D_6}(t_1, \dots, t_6) = \frac{1}{12}\left(t_1^6 + t_2^3 + 2t_3^2 + 2t_6 + 3t_1^2 t_2^2 + 3t_2^3\right)$$

The number of necklaces with three colours is

$$Z_{D_6}(3, \dots, 3) = \frac{1}{12}\left(3^6 + 3^3 + 2 \cdot 3^2 + 2 \cdot 3 + 3 \cdot 3^2 \cdot 3^2 + 3 \cdot 3^3\right) = 92$$

By Pólya's counting formula, the pattern inventory of necklaces is

$$P_{D_6}(w, r, b) = Z_{D_6}\left(w + r + b, w^2 + r^2 + b^2, w^3 + r^3 + b^3, w^4 + r^4 + b^4, w^5 + r^5 + b^5, w^6 + r^6 + b^6\right)$$

We are interested in the coeficient of $r^2 b^4$ *(two reds and four blues) and the coefficient of* $w^3 r b^2$ *(three whites, one red and two blues) in* $P_{D_6}(w, r, b)$. *In the following table we compute such coefficients according to each monomial on* $Z_{D_6}(t_1, \dots, t_6)$.

	r^2b^4	w^3rb^2
$(w+r+b)^6$	$\binom{6}{2}$	$\binom{6}{3}\binom{3}{2}$
$(w^2+r^2+b^2)^3$	$\binom{3}{2}$	0
$2(w^3+r^3+b^3)^2$	0	0
$2(w^6+r^6+b^6)$	0	0
$3(w+r+b)^2(w^2+r^2+b^2)^2$	$3\cdot(1+2)$	$3\cdot\binom{2}{1}\binom{2}{1}$
$3(w^2+r^2+b^2)^3$	$3\cdot\binom{3}{1}$	0

It follows that the coefficient of r^2b^4 is

$$\frac{1}{12}\left(\binom{6}{2}+\binom{3}{2}+3\cdot(1+2)+3\cdot\binom{3}{1}\right)=\frac{36}{12}=3,$$

and the coefficient of w^3rb^2 is

$$\frac{1}{12}\left(\binom{6}{3}\binom{3}{2}+3\cdot\binom{2}{1}\binom{2}{1}\right)=\frac{72}{12}=6.$$

6. (20 Pts.) Determine the number of ways to color the faces of an octahedron using the four colors red, blue, orange, and green, if
   (i) two colourings are considered to be equivalent if one can be obtained from the other by rotating the octahedron in some way.
   (ii) two colourings are considered to be equivalent if one can be obtained from the other by rotating the octahedron in some way and possibly exchanging red and orange, or blue and green, or both.

Solution: *The group of rotations of the octahedron is the same as the group of rotations of the cube. In fact, we can recover the action on the faces of the octahedron from the action on the vertices of the cube. Using the analysis that we did before for the cube, but considering the action on vertices we have:*

(a) The identity, of type $[1,1,1,1,1,1,1,1]$.

(b) 8 rotations of period 3 with axis through a pair of vertices. Those are or type $[3,3,1,1]$

(c) 6 half turns with axis through the midpoints of edges of type $[2,2,2,2]$.

(d) Two rotations of period 4 (one clockwise and its inverse) fixing a pair of faces; this gives us 6 elements of type $[4,4]$.

(e) One rotation of period 2 for each pair of faces, for a total of 3 elements of type $[2,2,2,2]$.

From here it follows that the cycle index of the permutation group G is

$$Z_G(t_1,\ldots,t_8)=\frac{1}{24}\left(t_1^8+8t_1^2t_3^2+6t_2^4+6t_4^2+3t_2^4\right)$$

(i) We know that the solution is given by evaluating $Z_G(t_1, \ldots, t_8)$ in $(4, \ldots, 4)$. Namely,

$$Z_G(4, \ldots, 4) = \frac{1}{24}\left(4^8 + 8 \cdot 4^2 4^2 + 6 \cdot 4^4 + 6 \cdot 4^2 + 3 \cdot 4^4\right) = \frac{69984}{24} = 2916$$

(ii) We need to find the number N of orbits of colouring under the action of both, G the permutation group induced by the rotations of the octahedron and H the permutation group on the colours. The permutation group H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. More precisely, the group H is

$$H = \{(r)(o)(g)(b), (r\ o)(g)(b), (r)(o)(g\ b), (r\ o)(g\ b)\}$$

We know that this number is given by

$$N = \frac{1}{|H|} \sum_{\tau \in H} Z_G(m_1(\tau), \ldots, m_n(\tau))$$

where $m_i(\tau) = \sum_{j|i} j \cdot z_j(\tau)$ and $z_j(\tau)$ denotes the number of cycles of length j in τ.

Since all the elements τ of H have only cycles of length 1 or 2, $z_j(\tau) = 0$ for every $j > 2$. It follows $m_i(\tau) = m_1(\tau)$ if i is odd and $m_i(\tau) = m_2(\tau)$ if i is even.

We compute $m_1(\tau)$ and $m_2(\tau)$ for every τ on the table below.

τ	$m_1(\tau)$	$m_2(\tau)$
$(r)(o)(g)(b)$	4	4
$(r\ o)(g)(b)$	2	$2+2$
$(r)(o)(g\ b)$	2	$2+2$
$(r\ o)(g\ b)$	0	$0+2\cdot 2$

The number N is given then by

$$N = \frac{1}{4}\big(Z_G(4,4,4,4,4,4,4,4) + 2Z_G(2,4,2,4,2,4,2,4) + Z_G(0,4,0,4,0,4,0,4)\big)$$

, that is

$$N = \frac{1}{4}\big(2916 + 2 \cdot 116 + 100\big) = 3132$$

To keep in mind: The group G of rotations of the cube has 24 elements, some of them fix a pair of opposite vertices, some of them fix a pair of opposite edges and some of them fix a pair of opposite faces. The group of roations of the octahedron is isomorphic to G. Moreover, the action of G on the vertices of one of the two polyhedra is equivalent to the action of G on the faces of the other.