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DISCRETE MATHEMATICS 2

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4. Symmetries of Graphs

Exercises.

4.1 Prove that there exists a graph on n vertices, all of them of valency k if and only if $n \geqslant k+1$ and nk is even.

4.2 Find the automorphism group of the cycle $\mathcal{C}_n$.

4.3 Consider the graph Γ whose vertices are the sets of size 2 of $\{1,2,3,4,5\}$ and so that $A \sim B$ if and only if $A \cap B = \emptyset$.
(a) Prove that Γ is isomorphic to the Petersen graph.
(b) Find the automorphism group of Γ.

4.4 Let H and K two subgroups of a group G. Assume that
- $H \triangleleft G$ and $K \triangleleft G$,
- $G = \langle H, K \rangle$,
- $H \cap K = \{1\}$.

Prove that $G \cong H \times K$.

4.5 Let H and K two groups and $\theta : K \to \mathrm{Aut}(H)$ and for $k \in K$ denote by θ_k the mapping $\theta(k) : H \to H$. Consider the set $H \times K$ and the operation $*_\theta$ given by $(h_1, k_1) *_\theta (h_2, k_2) = (h_1\theta_{k_1}(h_2), k_1k_2)$. Prove that the pair $(H \times K, *_\theta)$ is a group. This group is called the *(external) semidirect product of H and K with respect to θ* and is often denoted $H \rtimes_\theta K$.

4.6 Let H and K two subgroups of a group G. Assume that
- $H \triangleleft G$,
- $G = \langle H, K \rangle$,
- $H \cap K = \{1\}$.

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Prove that $G \cong H \rtimes_\theta K$, where $\theta_k(h) = khk^{-1}$. This is often called the *(internal) semidirect product of H and K* and it is usually denoted by $H \rtimes K$.

4.7 Prove that a complete bipartite graph $\mathcal{K}_{m,n}$ is vertex transitive if and only if $m = n$.

4.8 Find the automorphism group of the complete bipartite graph $\mathcal{K}_{m,n}$. Hint: consider the cases $n \neq m$ and $n = m$ separately.

4.9 Let Q_n denote the n-cube graph. Prove that

$$\mathrm{Aut}(Q_n) \cong \mathbb{Z}_2^n \rtimes_\theta S_n,$$

where $\theta(\sigma) : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is defined by

$$\theta(\sigma) : (v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}),$$

for every vector $(v_1, \ldots, v_n) \in \mathbb{Z}_2^n$. That is, θ is the natural action of S_n permuting the coordinates of $\mathbb{Z}_2^n$.

4.10 Assume that a graph Γ has m_1 connected components isomorphic to Δ_1, m_2 connected components isomorphic to Δ_2, etc. Such that $\Delta_i \cong \Delta_j$ if and only if $i = j$. Prove that the automorphism group of Γ is isomorphic to

$$(\mathrm{Aut}(\Delta_1)^{m_1} \times \cdots \times \mathrm{Aut}(\Delta_\ell)^{m_\ell}) \rtimes (S_{m_1} \times \cdots \times S_{m_\ell}).$$

4.11 The *cartesian product* of two graphs Γ and Δ is denoted by $\Gamma\Box\Delta$ and defined by $V(\Gamma\Box\Delta) = V(\Gamma) \times V(\Delta)$ and the edges of $\Gamma\Box\Delta$ are given by

$$(u_1, v_2) \sim (u_2, v_2) \Leftrightarrow \begin{cases} u_1 = u_2 \text{ and } v_1 \sim v_2 & \text{or} \\ v_1 = v_2 \text{ and } u_1 \sim u_2 \end{cases}$$

(i) Prove that if Γ and Δ are vertex transitive, then so it is $\Gamma\Box\Delta$.
(ii) Prove that $\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Delta) \leqslant \mathrm{Aut}(\Gamma\Box\Delta)$.
(iii) Give an example of two graphs Γ and Δ such that $\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(\Delta)$ is a proper subgroup of $\mathrm{Aut}(\Gamma\Box\Delta)$.

4.12 The *lexicographic product* of two graphs Γ and Δ is denoted by $\Gamma[\Delta]$ and defined by $V(\Gamma\Box\Delta) = V(\Gamma) \times V(\Delta)$ and the edges of $\Gamma\Box\Delta$ are given by

$$(u_1, v_2) \sim (u_2, v_2) \Leftrightarrow \begin{cases} u_1 \sim u_2 \text{ in } \Gamma & \text{or} \\ u_1 = u_2 \text{ and } v_1 \sim v_2 \text{ in } \Delta \end{cases}$$

Assume that Γ has m vertices. Observe that the action of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$ induces an action θ of $\mathrm{Aut}(\Gamma)$ on $(\mathrm{Aut}(\Delta))^m$.

(i) Prove that if Γ and Δ are vertex transitive, then so it is $\Gamma[\Delta]$.
(ii) Prove that the group

$$\mathrm{Aut}(\Delta)^m \rtimes_\theta \mathrm{Aut}(\Gamma)$$

is a subgroup of $\mathrm{Aut}(\Gamma[\Delta])$.

(iii) Give an example of two graphs Γ and Δ such that $\mathrm{Aut}(\Gamma)^m \rtimes_\theta \mathrm{Aut}(\Delta)$ is a proper subgroup of $\mathrm{Aut}(\Gamma[\Delta])$.

4.13 Prove that a graph is a Cayley graph if and only if its complement is a Cayley graph.

4.14 Prove that if p is an odd prime then there are only two groups of order $2p$, namely $\mathbb{Z}_{2p}$ and D_p. Hint: prove that $\mathbb{Z}_{2p} \cong \mathbb{Z}_p \times \mathbb{Z}_2$ and $D_p \cong \mathbb{Z}_p \rtimes \mathbb{Z}_2$. Use this to prove that the Petersen graph is not a Cayley graph.

4.15 Prove that there is no vertex-transitive graph of order at most 9 which is not a Cayley graph.

4.16 Prove that the cube graph is isomorphic to some generalised orbital graph of the symmetric group S_4.