Sets of real numbers

[We now drop AC, as we won't need it]

A subset $U \subseteq \mathbb{R}$ is <u>open</u> if for any $x \in U$ there exist $a < x < b$ such that $(a,b) \subseteq U$.

the open interval

(N.b., w.l.o.g. we can take $a, b \in \mathbb{Q}$.)

A subset $I \subseteq \mathbb{R}$ is an <u>interval</u> if whenever $a, b \in I$ where $a < b$, it holds that $(a,b) \subseteq I$.

<u>Proposition</u> A The following are equivalent for a subset $U \subseteq \mathbb{R}$.

1. U is open.

3. U is a disjoint union of countably many open intervals.

2. U is a (not necessarily disjoint) union of (countably many) intervals (p,q) where $p, q \in \mathbb{Q}$ $p < q$.

Proposition B The set $\mathcal{O}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$ of all open subsets has cardinality $2^{\aleph_0}$.

Proof Define $\boxed{\mathcal{I} := \{(p,q) \subseteq \mathbb{R} \mid p,q \in \mathbb{Q}, p<q\}}$

This is a countable set.

By 3. of Prop. A above, the function

$$F \longmapsto \bigcup_{n \in \mathbb{N}} F(n) \quad : \quad \mathcal{I}^{\mathbb{N}} \to \mathcal{O}(\mathbb{R})$$

is surjective.

$\boxed{\text{Even without AC}}$ this function has a section. $\boxed{\text{Exercise}}$

Hence $|\mathcal{O}(\mathbb{R})| \leq |\mathcal{I}^{\mathbb{N}}| = |\mathbb{N}^{\mathbb{N}}| = 2^{\aleph_0}$

Also $x \mapsto (-\infty, x) : \mathbb{R} \to \mathcal{O}(\mathbb{R})$ is injective.

Hence $2^{\aleph_0} = |\mathbb{R}| \leq \mathcal{O}(\mathbb{R})$.

$\square$

Proposition If $U \subseteq \mathbb{R}$ is open and $U \neq \emptyset$ then $|U| = 2^{\aleph_0}$.

Proof If U is open and nonempty then $(a,b) \subseteq U$ for $a < b \in \mathbb{R}$.

It is easy to construct a bijection from (a,b) to $\mathbb{R}$.

Hence $|(a,b)| = 2^{\aleph_0}$. So

$$2^{\aleph_0} = |(a,b)| \leq |U| \quad \text{because } (a,b) \subseteq U$$

$$|U| \leq |\mathbb{R}| = 2^{\aleph_0}$$

$\square$

A set $A \subseteq \mathbb{R}$ is <u>closed</u> if its complement $\mathbb{R} - A$ is open.

<u>Proposition</u> The set $\mathcal{A}(\mathbb{R})$ of all closed subsets of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.

<u>Proof</u> The map $X \mapsto \mathbb{R} - X$ gives a bijection $\mathcal{A}(\mathbb{R}) \longrightarrow \mathcal{O}(\mathbb{R})$. $\square$

$\boxed{\text{What cardinalities can closed subsets } A \subseteq \mathbb{R} \text{ have?}}$

- Every finite subset is closed.

  So every finite cardinality is a possibility.

- There also exist countably infinite closed subsets, e.g. $\mathbb{Z}$

  So $\aleph_0$ is a possibility.

- There also exist closed subsets of cardinality $2^{\aleph_0}$ e.g. $\mathbb{R}$, $[a,b]$

These are the only possibilities!

Theorem 1 If $A \subseteq \mathbb{R}$ is an uncountable closed set then $|A| = 2^{\aleph_0}$

So irrespective of CH in general, closed sets are sufficiently well-behaved that they cannot have a cardinality intermediate between $\aleph_0$ and $2^{\aleph_0}$.

Perfect sets (continued)

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Theorem 1 If $A \subseteq \mathbb{R}$ is an uncountable closed set then $|A| = 2^{\aleph_0}$

This is an immediate consequence of Theorems 2 + 3 below in combination.

Theorem 2 Every uncountable closed $A \subseteq \mathbb{R}$ contains a perfect subset

(closed sets have the perfect subset property)

Theorem 3 If A is perfect then $|A| = 2^{\aleph_0}$

Theorems 2 + 3 are of interest in their own right. They concern <u>perfect sets</u>, an important notion in the study of sets of real numbers.

Recall that $x \in \mathbb{R}$ is an accumulation point of $A \subseteq \mathbb{R}$ if, for any $\varepsilon > 0$, there $y \in A$ with $0 < |y - x| < \varepsilon$.

We write A' for the set of all accumulation points of A, and call A' the derived set of A.

Proposition

1. A set $A \subseteq \mathbb{R}$ is closed iff $A' \subseteq A$.
2. For any closed $A \subseteq \mathbb{R}$, it holds that A' is closed.

An element $x \in A \subseteq \mathbb{R}$ is said to be isolated if there exits $\varepsilon > 0$ s.t. there is no $y \in A$ with $0 < |y - x| < \varepsilon$

Clearly every $x \in A$ is either isolated or an accumulation point (but not both)

A set $A \subseteq \mathbb{R}$ is said to be perfect if it is nonempty, closed and has no isolated points

Obvious examples of perfect sets: $\mathbb{R}$, $[a,b]$, $(-\infty, a]$, $[a, \infty)$

An interesting example of a perfect set: the Cantor set

We recall some properties of compact sets.

$\boxed{\text{A subset } K \subseteq \mathbb{R} \text{ is } \underline{\text{compact}} \text{ if it is closed and bounded (there exist } a,b \in \mathbb{R} \text{ s.t. } K \subseteq (a,b)).}$

Proposition C

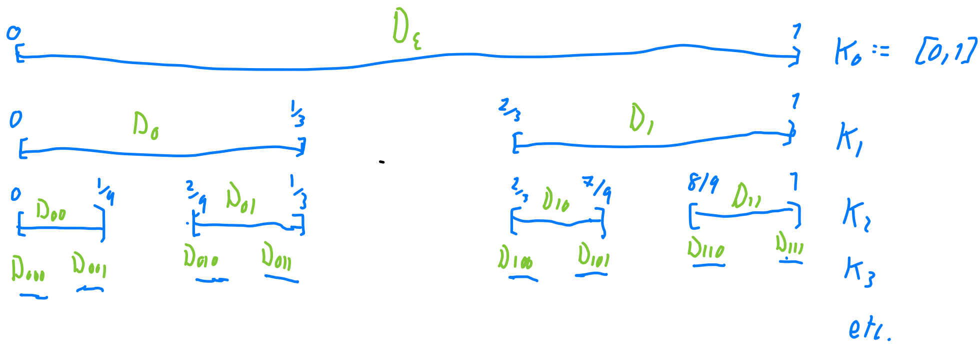
1. If $K \subseteq \mathbb{R}$ is compact then it attains its bounds.

   i.e. $\sup\{x \mid x \in K\} \in K$ (so we write $\max(K)$)

   $\inf\{x \mid x \in K\} \in K$ ($\min(K)$)

2. If $K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$ is a decreasing sequence of nonempty compact sets then $\bigcap_{n \in \mathbb{N}} K_n$ is also nonempty and compact.

The <u>Cantor set</u> is defined as follows



$$\boxed{\text{The } \underline{\text{Cantor set}} \text{ is } \quad K_\omega := \bigcap_{n\in\mathbb{N}} K_n}$$

More rigorously we define a collection of closed intervals D_w indexed by $w \in \{0,1\}^*$ ⟵ the set of finite sequences of 0's and 1's

the empty sequence

$$D_\varepsilon = [0,1]$$

$$\left.\begin{aligned} D_{w0} &= [a, \tfrac{2}{3}a + \tfrac{1}{3}b] \\ D_{w1} &= [\tfrac{1}{3}a + \tfrac{2}{3}b, b] \end{aligned}\right\} \quad \text{where } D_w = [a,b]$$

Define $K_n = \bigcup \{ D_w \mid |w| = n \}$ ($|w|$ = length of w)

Each K_n is nonempty and compact.

So $K_\omega := \bigcap_n K_n$ is nonempty and compact by Proposition C.

The Cantor set

Actually K_ω is <u>very</u> nonempty.

By definition

$$K_\omega = \bigcap_{n\in\mathbb{N}} \bigcup_{|w|=n} D_w$$

$$(*) \qquad = \bigcup_{s\in\{0,1\}^\omega} \bigcap_{w\prec s} D_w \qquad (w \prec s : w \text{ is a prefix of } s)$$

$\{0,1\}^\omega$ ← infinite sequences

But for each $s\in\{0,1\}^\omega$ $\bigcap_{w\prec s} D_w$ is nonempty by Proposition C(2)

In fact $\bigcap_{w\prec s} D_w$ is a singleton $\{c_s\}$ ← this defines the point c_s

And if $s\neq s'$ then $\left(\bigcap_{w\prec s} D_w\right)\cap\left(\bigcap_{w\prec s'} D_w\right) = \emptyset$

So $(*)$ shows that K_ω is a disjoint union of $2^{\aleph_0}$-many singletons. Hence has cardinality $2^{\aleph_0}$.

Finally K_ω has no isolated points. Consider any c_s in K_ω and $\varepsilon>0$. Let n be such that $3^{-n}<\varepsilon$. Let $s'\neq s$ be a sequence in $\{0,1\}^\omega$ that agrees with s on its first n bits.
Then $|c_s - c_{s'}| \leq 3^{-n} < \varepsilon$.
Since K_ω is nonempty, closed and has no isolated points, it is perfect.

Towards proof of Theorem 3.

<u>Lemma</u> If $A \subseteq \mathbb{R}$ is perfect and there exists $x \in A$ with $x > a$ then at least one of the sets $A \cap [a, \infty)$ and $A \cap (a, \infty)$ is perfect.

<u>Proof idea</u> One checks if $a \in A$ and a is not isolated in $A \cap [a, \infty]$ then $A \cap [a, \infty]$ is perfect. Otherwise $A \cap (a, \infty)$ is perfect. $\square$

Exercise – fill in the details!

Similarly if A is perfect and there is $x \in A$ with $a < b$ then $A \cap (-\infty, b]$ or $A \cap (-\infty, b)$ is perfect.

(the dual lemma)

Theorem 3 Every perfect set has cardinality $2^{\aleph_0}$.

Proof We need to show $|A| \geq 2^{\aleph_0}$ for perfect A.

We construct a family of bounded perfect subsets of A
$(A_w)_{w \in \{0,1\}^*}$

A_ε := some bounded perfect subset of A
(this exists by the lemma and its dual)

To define A_{w0} and A_{w1} from A_w:

Suppose $\min(A_w) = a$ and $\max(A_w) = b$

Define

$$A_{w0} := \begin{cases} A_w \cap (-\infty, \frac{2}{3}a + \frac{1}{3}b] & \text{if perfect} \\ A_w \cap (-\infty, \frac{2}{3}a + \frac{1}{3}b) & \text{otherwise} \end{cases}$$

$$A_{w1} := \begin{cases} A_w \cap [\frac{1}{3}a + \frac{2}{3}b, \infty) & \text{if perfect} \\ A_w \cap (\frac{1}{3}a + \frac{2}{3}b, \infty) & \text{otherwise} \end{cases}$$

For any $s \in \{0,1\}^\omega$

$\bigcap_{w \prec s} A_w$ is the intersection of a decreasing sequence of nonempty compact sets, hence nonempty and compact. Easily shown to be a singleton.

If $s \neq s' \in \{0,1\}^\omega$ then $\left(\bigcap_{w \prec s} A_w\right) \cap \left(\bigcap_{w \prec s'} A_w\right) = \emptyset$

So $\bigcup_{s \in \{0,1\}^\omega} \bigcap_{w \prec \alpha} A_w$ is a subset of A

given as a disjoint union of $2^{\aleph_0}$-many singletons.

So indeed $2^{\aleph_0} \leq |A|$. $\square$

Theorem 2 Every uncountable closed subset of $\mathbb{R}$ has a perfect subset.

Proof Suppose A is uncountable and closed.
We define a transfinite sequence of closed subsets of A (by transfinite recursion)

$$A_0 := A$$

$$A_{\alpha+1} := (A_\alpha)' \quad \leftarrow \text{the derived set}$$

$$A_\lambda := \bigcap_{\alpha<\lambda} A_\alpha \qquad (\lambda \text{ a limit ordinal})$$

By transfinite induction:

- Every A_α is indeed closed
- $\alpha \leq \beta \implies A_\alpha \supseteq A_\beta$

It cannot be the case that $A_\alpha \neq A_\beta$ whenever $\alpha \neq \beta$ because this would give us an injection $\underline{Ord} \to \mathcal{A}(\mathbb{R})$ contradicting that $\mathcal{A}(\mathbb{R})$ is a set.

set of closed subsets

So there exist $\alpha < \beta$ with $A_\alpha = A_\beta$.
Hence $A_\alpha \supseteq A_{\alpha+1} \supseteq A_\beta = A_\alpha$; i.e. $A_\alpha = A_{\alpha+1}$.

Let γ be the smallest ordinal s.t. $A_\gamma = A_{\gamma+1}$.

Since $A_\gamma = A_{\gamma+1} = (A_\gamma)'$, A_γ has no isolated points.

So either A_γ is empty or perfect.

It remains to show A_γ is nonempty.

Define $F := A - A_\gamma$. We show that F is countable.

For any $c \in F$ there exists a unique α with $c \in A_\alpha - A_{\alpha+1}$. As c is isolated in A_α there exists a rational interval (p,q) with $c \in (p,q)$ and $(p,q) \cap A_\alpha = \{c\}$.

Let $f: F \to \mathcal{I}$ (defined at start of lecture) be a function sending c to some (p,q) as above.

(This can be defined without choice.)

We show that f is injective.

Suppose $f(c) = f(c')$ where $c \in A_\alpha - A_{\alpha+1}$
and $c' \in A_\beta - A_{\beta+1}$ and $\alpha \leq \beta$.

Then $c' \in f(c') = f(c)$ and $c' \in A_\beta \subseteq A_\alpha$.

So $c' \in f(c) \cap A_\alpha = \{c\}$; i.e. $c' = c$.

So f is indeed injective.

Since $f : F \to \mathcal{I}$ is injective and $\mathcal{I}$ is countable,
F is countable.

Since $|A|$ is uncountable and F is countable,
$|A| - F$ is nonempty. I.e. A_γ is nonempty,
as required.

$\square$

It follows from the proof that $\gamma < \omega_1$.

Exercise.