For this lecture we assume Countable choice CC

Borel sets, games and determinacy

The open subsets of $\mathbb{R}$ are closed under finite intersections and countable (indeed arbitrary) unions. Open sets are not closed under countable intersections.

Similarly, closed subsets of $\mathbb{R}$ are closed under finite union and countable (indeed arbitrary) intersection. They are not closed under countable union.

A set $X \subseteq \mathbb{R}$ is said to be

F_σ if it is a countable union of closed sets

G_δ if it is a countable intersection of open sets

Exercise find interesting examples of F_σ and G_δ sets that are neither open nor closed.

The F_σ sets contain every open set (exercise) and are closed under finite intersections and countable (but not arbitrary) unions. But they are not closed under countable intersections.

The G_δ sets are the complements of F_σ sets, so ... (dualise the above paragraph).

The Borel hierarchy

We iterate the process of closing families of sets under countable unions and intersections into the transfinite.

Specifically we define families

$$\underline{\Sigma}^0_\alpha \qquad \underline{\Pi}^0_\alpha$$

fixed at 0 for the Borel hierarchy

bold face in books

an ordinal index

with $\underline{\Sigma}^0_\alpha, \underline{\Pi}^0_\alpha \subseteq \mathcal{P}(\mathbb{R})$

We shall use the simpler notation Σ_α, Π_α

The hierarchy $\Sigma_\alpha, \Pi_\alpha$ are defined by transfinite recursion on $\alpha \geq 1$

$\Sigma_1 := \mathcal{O}(\mathbb{R})$ the set of open sets

$\Pi_1 := \mathcal{A}(\mathbb{R})$ the set of closed sets

$$\Sigma_{\alpha+1} := \left\{ B \subseteq \mathbb{R} \;\middle|\; B = \bigcup_{n=0}^{\infty} B_n \text{ where each } B_n \in \Pi_\alpha \right\}$$

$$\Pi_{\alpha+1} := \left\{ B \subseteq \mathbb{R} \;\middle|\; B = \bigcap_{n=0}^{\infty} B_n \text{ where each } B_n \in \Sigma_\alpha \right\}$$

α can depend on n

$$\Sigma_\lambda := \left\{ B \subseteq \mathbb{R} \;\middle|\; B = \bigcup_n B_n \text{, where for every } n \text{ we have } B_n \in \Pi_\alpha \text{ for some } \alpha < \lambda \right\}$$

$$\Pi_\lambda := \left\{ B \subseteq \mathbb{R} \;\middle|\; B = \bigcap_n B_n \text{, where for every } n \text{ we have } B_n \in \Sigma_\alpha \text{ for some } \alpha < \lambda \right\}$$

(λ a limit ordinal)

Proposition A

(1) $\Sigma_\alpha \subseteq \Pi_{\alpha+1}$, $\Pi_\alpha \subseteq \Sigma_{\alpha+1}$

(2) $\alpha \leq \beta \Rightarrow \Sigma_\alpha \subseteq \Sigma_\beta$ and $\Pi_\alpha \subseteq \Pi_\beta$

(3) Σ_α is closed under finite intersections and countable unions

Π_α " " unions " " intersections

(4) $B \in \Sigma_\alpha \iff \mathbb{R} - B \in \Pi_\alpha$

(5) $\Sigma_{\omega_1} = \Pi_{\omega_1} = \Sigma_{\omega_1+1} = \Pi_{\omega_1+1}$

[In fact ω_1 is the smallest ordinal for which (5) holds. This is tricky to prove.]

Proof We just prove statement (5).

(The rest is easier — exercise.)

To show $\Pi_{\omega_1} \subseteq \Sigma_{\omega_1}$ consider any $B \in \Pi_{\omega_1}$.

Then $B = \bigcap_n B_n$ where $B_n \in \Sigma_{\alpha_n}$ for some countable α_n.

Define $\beta := \sup_n \alpha_n$. This is countable.

By (2) every $B_n \in \Sigma_\beta$.

Therefore $B \in \Pi_{\beta+1} \subseteq \Sigma_{\beta+2}$ by (1).

$\subseteq \Sigma_{\omega_1}$ by (2) as $\beta+2$ is countable.

So indeed $\Pi_{\omega_1} \subseteq \Sigma_{\omega_1}$.

The converse $\Sigma_{\omega_1} \subseteq \Pi_{\omega_1}$ is proved similarly.

Thus indeed $\Sigma_{\omega_1} = \Pi_{\omega_1}$.

The other two equalities then follow easily from the definitions of Σ_{ω_1+1} and Π_{ω_1+1}. $\square$

Borel sets

Usual definition in maths

The family $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ of <u>Borel</u> sets is the smallest $\mathcal{B}$ that:

- contains every open set
- and is closed under complement and countable unions.

Why does such a $\mathcal{B}$ exist?

Consider the collection $\Phi \subseteq \mathcal{P}\mathcal{P}(\mathbb{R})$ defined by

$$\Phi = \{ \mathcal{D} \subseteq \mathcal{P}(\mathbb{R}) \mid \mathcal{D} \text{ contains every open set and is closed under complements and countable unions} \}.$$

Easily:

- $\Phi \neq \emptyset$ because $\mathcal{P}(\mathbb{R}) \in \Phi$
- If $\Psi \subseteq \Phi$ then $\bigcap \Psi \in \Phi$

Therefore $\bigcap \Phi \in \Phi$ and is the smallest set in Φ.

Hence $\mathcal{B} := \bigcap \Phi$.

Theorem $\mathcal{B} = \Sigma_{\omega_1} = \Pi_{\omega_1}$

Proof Statements (2), (4) + (5) of Proposition A say that Σ_{ω_1} contains all open sets and is closed under countable unions and complements. Hence $\mathcal{B} \subseteq \Sigma_{\omega_1}$.

For the converse one proves by transfinite induction on α that $\Sigma_\alpha, \Pi_\alpha \subseteq \mathcal{B}$ for all ordinals α.

This is routine. $\boxed{\text{Exercise}}$ $\square$

The families Σ_α and Π_α *stratify* the Borel sets into a transfinite hierarchy, the *Borel hierarchy* of height ω_1.

This allows us to prove properties of Borel sets.

Proposition (AC) $|\mathcal{B}| = 2^{\aleph_0}$

$$\left(\text{cf. } |\mathcal{P}(\mathbb{R})| = 2^{2^{\aleph_0}} \right)$$

Proof $\mathcal{O}(\mathbb{R}) \subseteq \mathcal{B}$ so $2^{\aleph_0} \leq |\mathcal{B}|$

For the converse inequality we first show $|\Sigma_\alpha| \leq 2^{\aleph_0}$ by transfinite induction on $\alpha \leq \omega_1$

In the case of a successor ordinal $\alpha+1$:

we have a surjection

$$(B_n)_{n\in\mathbb{N}} \mapsto \bigcup_n B_n \quad : \quad (\Pi_\alpha)^{\mathbb{N}} \longrightarrow \Sigma_{\alpha+1}$$

by AC from above

$$\text{So } \quad |\Sigma_{\alpha+1}| \leq |\Pi_\alpha|^{\aleph_0}$$

$$\leq \left(2^{\aleph_0}\right)^{\aleph_0} \qquad \text{by ind. hyp.}$$

$$= 2^{\aleph_0}$$

For a limit ordinal λ

cardinal sum

$$\left|\bigcup_{\alpha<\lambda} \Sigma_\alpha\right| \leq \sum_{\alpha<\lambda} |\Sigma_\alpha| \leq \sum_{\alpha<\lambda} 2^{\aleph_0} \quad \text{ind. hyp.}$$

Sigma α in Borel hierarchy

$$= \lambda \cdot 2^{\aleph_0} \leq 2^{\aleph_0} \quad \text{because } \lambda \leq \omega_1 \underset{(AC)}{\leq} 2^{\aleph_0}$$

So also $\left|\bigcup_{\alpha<\lambda} \Pi_\alpha\right| \leq 2^{\aleph_0}$

So again we have a surjection

$$(B_n)_{n\in\mathbb{N}} \mapsto \bigcup_n B_n : \left(\bigcup_{\alpha<\lambda} \Pi_\alpha\right)^{\mathbb{N}} \to \Sigma_\lambda$$

So $|\Sigma_\lambda| \leq \left|\bigcup_{\alpha<\lambda} \Pi_\alpha\right|^{\aleph_0}$ (AC)

$$\leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}. \qquad \square$$

Borel sets, games & determinacy (continued)

The notion of Borel set makes sense for *any* topological. However, the Borel sets have really good properties for a particularly nice class of spaces: the *Polish spaces*.

A *Polish space* is a topological space that is

- completely metrizable

  i.e., the topology arises from a metric with respect to which the space is Cauchy complete (every Cauchy sequence has a limit)

- separable

  there exists a countable dense subset

$\mathbb{R}$ is an obvious example of a Polish space

Another important Polish space : Baire space

Baire space is $\mathbb{N}^{\mathbb{N}}$

$s = s_0 s_1 s_2 s_3 s_4 \ldots$

$t = t_0 t_1 t_2 t_3 t_4 \ldots$

with the following metric

$$d(s,t) = \begin{cases} 0 & \text{if } s = t \\ 2^{-n} & \text{otherwise where } n \in \mathbb{N} \text{ is the least such that } s_n \neq t_n \end{cases}$$

We shall use Baire space as an "arena" for playing games.

Gale - Stewart games

Two players I and II take it in turns to pick natural numbers

I	n_0		n_2		n_4		....
II		n_1		n_3		n_5	...

They play to generate an infinite sequence $(n_i)_{i \in \mathbb{N}}$.

Every subset $G \subseteq \mathbb{N}^{\mathbb{N}}$ determines a game in which G is the <u>goal</u> (or <u>payoff</u>) set for player I.

Player I <u>wins</u> the play $(n_i)_{i \in \mathbb{N}}$ if $(n_i)_{i \in \mathbb{N}} \in G$.

Player II <u>wins</u> the play if $(n_i)_{i \in \mathbb{N}} \notin G$

For any $X \subseteq \mathbb{R}$ we define a Gale-Stewart game Perf_X as follows.

Define $A_0 = \mathbb{R}$

On round $n \geq 1$

Player I chooses $p < q < p' < q' \in \mathbb{Q}$

such that $[p, q] \cup [p', q'] \subseteq A_{n-1}$

and $\max(q-p, q'-p') < 1/2^n$

Player II chooses either 0 or 1

In case of 0 define $A_n := [p, q]$

1 $A_n := [p', q']$

Player I wins if $\{x\} := \bigcap_n A_n$ satisfies $x \in X$

Player I's moves are of the form

(p, q, p', q')

Such a move can be encoded as a natural number as $\mathbb{Q}$ is countable.

If X is a countable set then Player II has a "winning strategy" for perf$X$ as follows

Write X as $\{x_0, x_1, x_2, x_3, \ldots\}$

On round n Player II chooses 0 or 1 such that $x_{n-1} \notin A_n$

If X contains a perfect subset then Player I has a "winning strategy" for perf$X$ as follows

Let $P_0 \subseteq X$ be perfect.

On round n, Player I chooses $p < q < p' < q'$ so that $[p, q]$ contains a perfect subset $L \subseteq P_{n-1}$ and $[p', q']$ " " " " $R \subseteq P_{n-1}$

(This is done by versions of lemmas from last week)

If Player II chooses 0, define $P_n := L$

1 $\quad P_n := R$

Player I wins because $\bigcap_n A_n = \bigcap_n P_n \in X$.

A strategy for Player I is a function

For a general Gale-Stewart game

$$\sigma_I : \{ w \in \mathbb{N}^* \mid |w| \text{ is even} \} \to \mathbb{N}$$

A strategy for Player II is a function

$$\sigma_{II} : \{ w \in \mathbb{N}^* \mid |w| \text{ is odd} \} \to \mathbb{N}$$

A play $n_0 n_1 n_2 n_3 \ldots$ follows σ_I if

for all i

$$n_{2i} = \sigma_I (n_0 n_1 \ldots n_{2i-1})$$

In particular $n_0 = \sigma_I(\varepsilon)$ the empty sequence

It follows σ_{II} if

$$n_{2i+1} = \sigma_{II} (n_0 n_1 \ldots n_{2i})$$

σ_I is a winning strategy (from w) for Player I

if every play (with prefix w) that follows σ_I is

winning for Player I.

σ_{II} is a winning strategy if ... (as above)

Proposition For any goal set $G \subseteq \mathbb{N}^{\mathbb{N}}$, it is not the case that both players simultaneously have winning strategy.

Definition A goal set G is determined if one of PI and PII has a winning strategy.

Theorem (Borel determinacy) If $G \subseteq \mathbb{N}^{\mathbb{N}}$ is a Borel set then G is determined.

This is an important theorem due to Martin, The proof is beyond this course.

Below, we prove the following weaker theorem.

Theorem If $G \subseteq \mathbb{N}^{\mathbb{N}}$ is closed then G is determined.

This proof is not examinable.

Proof Let G be a closed goal set.

Define $\sigma_I(w)$ (where $|w|$ is even) by:

$$\sigma_I(w) = \begin{cases} \text{the smallest } n \text{ such that Player II does not have a winning strategy from } wn, \text{ if such an } n \text{ exists} \\ 0 \quad \text{otherwise} \end{cases}$$

We prove that if II does not have a winning strategy then σ_I is a winning strategy for I.

Suppose II does not have a winning strategy.

Let $s = s_0 s_1 s_2 \cdots$ be any play following σ_I.

We must show that $s \in G$.

We claim that for every $K \geq 0$, II does not have a winning strategy from $s_0 s_1 \cdots s_{K-1}$.
By induction on K.

<u>$K=0$</u> $s_0 s_1 \cdots s_{K-1} = \varepsilon$. We are at the start of the game.
Indeed II does not have a winning strategy by assumption.

<u>$K>0$ and K is odd</u>

Then $s_{K-1} = \sigma_{I}(s_0 \cdots s_{K-2})$ = the smallest n such that PII does not have a winning strategy from $s_0 \cdots s_{K-2} n$
because such an n exists, by the induction hypothesis (PII does not have a winning strategy from $s_0 \cdots s_{K-2}$)
using

If $|w|$ is even then PII has a winning strategy from w iff PII has a winning strategy from wn for all n

$k > 0$ and k is even

By induction hypothesis II has no winning strategy from $s_0 \dots s_{k-2}$. So II has no winning strategy from $s_0 \dots s_{k-1}$ using

If $|w|$ is odd then II has a winning strategy from w iff II has a winning strategy from wn for some n.

We have shown that for every $k \geq 0$ II does not have a winning strategy from $s_0 \dots s_{k-1}$.
So there must be some continuation play $t^{(k)}$ such that $s_0 \dots s_{k-1} t^{(k)} \in G$ (i.e. II loses)

s_0 $\quad t^{(0)}$

$s_0 t^{(1)}$

$s_0 s_1 t^{(2)}$

$s_0 s_1 s_2 t^{(3)}$

$s_0 s_1 s_2 s_3 t^{(4)}$ is a Cauchy sequence in G.

As G is closed the limit of this Cauchy sequence also belongs to G. This limit is $s_0 s_1 s_2 s_3 s_4 \dots\dots$.

Indeed $s \in G$. $\square$