Expressing Mathematics in English **Name:** ____________________
Written exam **Enrolment no.:** ____________________
5 June, 2019
Time Limit: 150 Minutes

1. The first six triangular numbers are: one, three, six, ten, fifteen, twenty-one. The *ordinal* numbers corresponding to these six triangular numbers are:

   first, ____

2. Define in words what it means for a triangle to be an isosceles triangle and what it means for a triangle to be an equilateral triangle. Illustrate your answer with an accompanying picture of each type of triangle.

3. Write down the names of five regular polygons and the number of edges each one has.

   One answer has been filled in for you.

equilateral triangle	$n = 3$
________________	$n =$ ____
________________	$n =$ ____
________________	$n =$ ____
________________	$n =$ ____
________________	$n =$ ____

4. Any number of the form $a + bi$ where a and b are real and $i = \sqrt{-1}$ is called a ________________ number. A complex number of the form $a + bi$ where $b = 0$ is called ________________. A complex number of the form bi where $b \neq 0$ is called ________________. Let $z = a + bi$ be a complex number. The real number a is called the ________________ of z and the real number b is called the ________________ of z. The complex number $a - bi$ is called the ________________ of z. It can be obtained by ________________ z in the real axis. The ________________ of z is $\sqrt{a^2 + b^2}$. The fundamental theorem of ________________ states that every nonconstant ________________ has a complex ________________ .

5. Decide whether the following statements are true (T) or false (F).

   (a) Every real number is either rational or irrational.

   (b) Every integer is a natural number.

   (c) Every prime number less than 100 is odd.

   (d) There exists a largest prime number.

   (e) The square root of 2 is irrational.

   (f) The square of an integer x is greater than or equal to x.

   (g) The product of two rational numbers is rational.

   (h) The hypotenuse is the longest side of a right-angled triangle.

   (i) The trigonometric functions $\sin(x)$ and $\cos(x)$ are twice continuously differentiable.

   (j) Two vectors $u, v \in \mathbb{R}^n$ are orthogonal if and only if $u \times v = 0$.

6. (a) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Write down formulas for

   1. The transpose of A

   2. The determinant of A

   3. The inverse of A

   (b) Write down the property that characterises the inverse of a square matrix A.

7. Translate the following into English.

Naj bo $A : U \to V$ linearna preslikava. Izberimo bazi $\mathscr{B} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ za U in $\mathscr{C} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ za V. Po izreku 1.7 je A natanko določena, če poznamo slike baznih vektorjev $A\mathbf{u}_1, A\mathbf{u}_2, \dots, A\mathbf{u}_n$. Razvijmo te vektorje po bazi $\mathscr{C}$:

$$\begin{aligned} A\mathbf{u}_1 &= \alpha_{11}\mathbf{v}_1 + \alpha_{21}\mathbf{v}_2 + \cdots + \alpha_{m1}\mathbf{v}_m \\ A\mathbf{u}_2 &= \alpha_{12}\mathbf{v}_1 + \alpha_{22}\mathbf{v}_2 + \cdots + \alpha_{m2}\mathbf{v}_m \\ &\vdots \\ A\mathbf{u}_n &= \alpha_{1n}\mathbf{v}_1 + \alpha_{2n}\mathbf{v}_2 + \cdots + \alpha_{mn}\mathbf{v}_m \end{aligned}$$

Koeficienti razvoja tvorijo matriko:

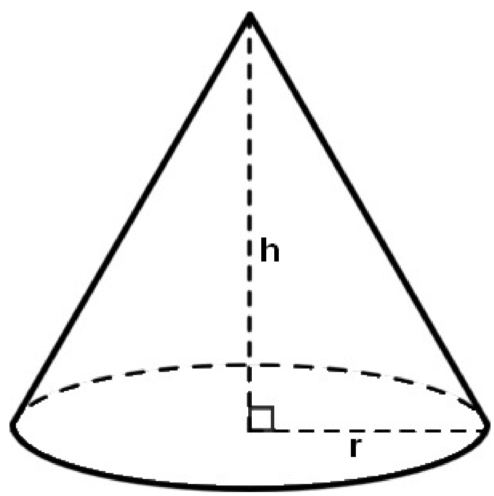
$$A_{\mathscr{B}\mathscr{C}} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{mn} \end{bmatrix}.$$

To matriko imenujemo *matrika prirejena linearni preslikavi* glede na bazi $\mathscr{B}$ in $\mathscr{C}$.

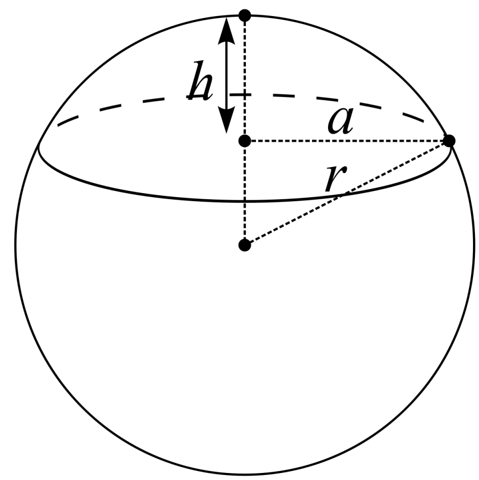
8. Connect each property of the order relation on $\mathbb{R}$ to its corresponding mathematical definition.

Antisymmetry	Either $a > b$, $a = b$, or $a < b$
Reflexivity	$a \leq a$
Order preserved by sum	If $a \leq b$ and $b \leq c$, then $a \leq c$
Transitivity	If $a \leq b$, then $a + c \leq b + c$.
Trichotomy	$a \leq b$ and $b \leq a$ implies $a = b$
Preservation under scalar multiplication	$a \leq b$ implies $ca \leq cb$ for all $c > 0$.

9. Consider the shapes below.



L



R

(a) What is shape L called?

(b) What is shape R called?

(c) What is the value r in shape R called?

(d) What is the value h in shape L called?

(e) What quantity relating to L does the formula $\frac{1}{3}\pi r^2 h$ compute?

(f) What quantity relating to R does the formula $4\pi r^2$ compute?

10. Fill in the blanks using each of the following phrases precisely once.

integral from 0 to 1 derivative at 0 limit as $x \to 0$

absolute value at $x = 0$ value at $x = 0$

The ______________________________ of $f(x) = x^2 - x$ is -1.

The ______________________________ of $f(x) = x$ is $\frac{1}{2}$.

The ______________________________ of $f(x) = (x+1)^3$ is 1.

The ______________________________ of $f(x) = \begin{cases} 3 & \text{if } x \neq 0 \\ -2 & \text{if } x = 0 \end{cases}$ is 2.

The ______________________________ of $f(x) = \begin{cases} 3 & \text{if } x \neq 0 \\ -2 & \text{if } x = 0 \end{cases}$ is 3.

11. Consider the formula below, associated to a data set $\{x_1, \ldots, x_n\}$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}.$$

(a) What is the name of the value μ

(b) What is the name of the value σ

(c) Compute μ for the data set $\{0, 3, 6, 4, 4, 1\}$.

(d) Compute σ for the same data set.

12. Consider the following algorithm for natural numbers $n_1 \geq n_2 > 0$.

    1. Let r be the remainder when n_1 is divided by n_2.
    2. If $r = 0$, then output n_2.
    3. Else go back to Step 1 with (n_2, r) in place of (n_1, n_2).

    (a) What is this algorithm called?

    (b) What is it used for?

    (c) Apply the above algorithm with $n_1 = 2311$ and $n_2 = 23$.

13. Read the following proof and then answer the multiple choice questions below.

Theorem. If $f : [a, b] \to \mathbb{R}$ is a continuous function with $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof. We may prove this theorem in the following steps.

1. Note that the set S of all points $x \in [a, b]$ at which $f(x)$ is negative is nonempty.
2. Therefore, this set has a least upper bound $c \in \mathbb{R}$.
3. Observe that the real number c must be strictly greater than a.
4. Also, observe that the real number c must be strictly less than b.
5. The value of f at c cannot be greater than 0 by continuity.
6. The value of f at c cannot be less than 0 by the definition of c and the continuity of f.
7. Therefore, by the trichotomy axiom it must be the case that $f(c) = 0$. □

(a) Why is the set referred to in the first step nonempty?

(A) Because the right endpoint of the interval domain belongs to it.
(B) Because the left endpoint of the interval domain belongs to it.
(C) Because the endpoints of the interval domain belong to it.
(D) Because every continuous function has a positive and negative part.

(b) How do we know that there exists a real number c as introduced in step 2?

(A) Because the supremum of a subset of the real numbers is unique.
(B) Because every subset of the real numbers has a least upper bound.
(C) Because the set is bounded and the real numbers enjoy the completeness axiom.
(D) Because we can set $c = b$ for instance.

(c) The assertion in step 3 is true for the following reason.

(A) The real number c is in the interval $[a, b]$ by definition.
(B) The function f is continuous and $f(b) > 0$.
(C) The function f is continuous and $f(a) < 0$.
(D) The real number b is an upper bound for this set and $b > a$.

(d) The assertion in step 4 is true for the following reason.

(A) The real number c is in the interval $[a, b]$ by definition

(B) The function f is continuous and $f(b) > 0$.

(C) The function f is continuous and $f(a) < 0$.

(D) The real number $(b - a)/2$ is an upper bound for S that is strictly less than b.

(e) From step 3 and step 4 we conclude the following.

(A) The real number c^2 is greater than a^2 and less than b^2.

(B) The real number c belongs to the interval (a, b).

(C) The value of f at c is strictly greater than a.

(D) The value of f at c is strictly less than b.

(f) The author is asked how they used continuity in step 5. Which of the following is a correct response?

(A) There is a sequence of points $(x_n)_{n\in\mathbb{N}}$ in $[a, b]$ converging to c such that $f(x_n)$ is negative for each $n \in \mathbb{N}$, and so by continuity the limit $f(c) = \lim_{n\to\infty} f(x_n)$ is negative too.

(B) This is an immediate consequence of the intermediate value theorem.

(C) If $f(c) > 0$ then $(f(c)-f(x))/(c-x) > f(c)/(c-x)$ for x to the left of c. Hence the derivative, which we may calculate by taking the limit of $(f(c)-f(x))/(c-x)$ as x approaches c from the left, is infinite. This contradicts f being continuous.

(D) If $f(c) > 0$ then by continuity there exists $\delta > 0$ such that $f(x) > f(c)/2$ whenever $x \in [a, b]$ and $c - \delta < x < c + \delta$. This is a contradiction.

(g) The author is asked to explain step 6. Which of the following is a correct response?

(A) If $f(c) < 0$ then $(f(c) - f(x))/(c - x) > f(c)/(c - x)$ for x to the right of c. Hence the derivative, which we may calculate by taking the limit of $(f(c) - f(x))/(c - x)$ as x approaches c from the right, is infinite. This contradicts f being continuous.

(B) If $f(c) < 0$ then since f is continuous on $[c, b]$ the intermediate value theorem tells us there exists a $c' \in (c, b)$ such that $f(c') = 0$, and we are done. Hence we suppose this is not the case.

(C) If $f(c) < 0$ then by continuity there exists $\delta > 0$ such that $f(x) < 0$ whenever $x \in [a, b]$ and $c-\delta < x < c+\delta$. This contradicts c being an upper bound of S.

(D) There is a sequence of points $(x_n)_{n\in\mathbb{N}}$ in $[a, b]$ converging to c such that $f(x_n)$ is positive for each $n \in \mathbb{N}$, and so by continuity the limit $f(c) = \lim_{n\to\infty} f(x_n)$ is positive too.

(h) Clearly if we replace $f(a) < 0$ by $f(a) > 0$ the theorem is no longer true. What is the first step of the proof that fails in this case?

(A) Step 1

(B) Step 2

(C) Step 3

(D) Step 5

(i) The following is also true. If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) < f(b)$ and $d \in (f(a), f(b))$, then there exists $c \in (a, b)$ such that $f(c) = d$. How could we obtain this result from the one proven above?

(A) We could simply apply the above theorem to the continuous function $g(x) = f(x - c)$.

(B) We could simply apply the above theorem to the continuous function $g(x) = f(x) - d$.

(C) Since d is a maximum of f, we know that $f(c) = d$ implies that the derivative of f at c is zero. Therefore we want to find zeros of the derivative function and this may be done using the above theorem.

(D) Since the function $0 \cdot f$ has a zero at c, we may apply the above theorem to find it.