

Kardinalna Aritmetika: Izpit

June 30, 2020

This exam contains four questions. You are required to answer all four. All answers must be justified. You have 120 minutes to complete the exam. You may answer in English or in Slovene. Good luck!

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Seat (3.04)

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Question 1 (25 marks)

a) (6 marks) Given a class $\underline{X}$ and a class $\underline{Y}$ define what it means for a subclass $\underline{F} \subseteq \underline{X} \times \underline{Y}$ to be the graph of an injective function from $\underline{X}$ to $\underline{Y}$.

b) (7 marks) Show that if $\underline{X}$ is a set then the graph $\underline{F} \subseteq \underline{X} \times \underline{Y}$ of an injective function is a set.

c) (3 marks) Show that if $\underline{Y}$ is a set then the graph $\underline{F} \subseteq \underline{X} \times \underline{Y}$ of an injective function is a set.

d) (9 marks) Show that if $\underline{Y}$ is a set then the class of all injective functions from $\underline{X}$ to $\underline{Y}$ is a set.

Question 2 (25 marks)

a) (6 marks) Define the product well-order $(X, <_X) \times (Y, <_Y)$, where $(X, <_X)$ and $(Y, <_Y)$ are well-orders.

b) (7 marks) Prove that $(X, <_X) \times (Y, <_Y)$ is indeed a well-order.

c) (6 marks) Given arbitrary well-orders $(X, <_X)$, $(Y, <_Y)$ and $(Z, <_Z)$, does it follow that the well-orders

$$(X, <_X) \times ((Y, <_Y) + (Z, <_Z)) \quad \text{and} \quad ((X, <_X) \times (Y, <_Y)) + ((X, <_X) \times (Z, <_Z))$$

are order isomorphic? Either define an explicit order isomorphism or find a counterexample.

d) (6 marks) Given arbitrary well-orders $(X, <_X)$, $(Y, <_Y)$ and $(Z, <_Z)$, does it follow that the well-orders

$$((X, <_X) + (Y, <_Y)) \times (Z, <_Z) \quad \text{and} \quad ((X, <_X) \times (Z, <_Z)) + ((Y, <_Y) \times (Z, <_Z))$$

are order isomorphic? Either define an explicit order isomorphism or find a counterexample.

Question 3 (25 marks)

a) (10 marks) Suppose X has cardinality $\aleph_0$ and Y is a countably infinite subset of X . What are the possible cardinalities of the set difference $X - Y$?

b) (10 marks) Suppose X has cardinality $2^{\aleph_0}$ and Y is a countably infinite subset of X . What are the possible cardinalities of the set difference $X - Y$? If you make use of any theorem from the lectures then you should also prove the theorem.

(More space for your solution to part (b)).

c) (5 marks) A real number x is said to be *algebraic* if it satisfies some equation $p(x) = 0$ where p is a polynomial with rational coefficients. A real number that is not algebraic is said to be *transcendental*. What is the cardinality of the set of transcendental numbers?

Question 4 (25 marks)

In the space below and on the opposite page, answer **exactly one** of the two questions below.

1.
 - (a) Recall the definition of the *cumulative hierarchy* $(V_\alpha)_{\alpha \in \text{Ord}}$. (5 marks)
 - (b) Defining $\underline{V} := \bigcup_\alpha V_\alpha$, prove that every element $x \in \underline{V}$ is a set. (6 marks)
 - (c) Define the *rank* of any set $X \in \underline{V}$ to be the smallest ordinal α such that $X \in V_\alpha$. Explain why this notion of rank is well-defined. (2 marks)
 - (d) Define a class function $\underline{F}: \text{Ord} \rightarrow \underline{V}$ such that, for every ordinal α , the rank of $\underline{F}(\alpha)$ is $\alpha + 1$. (10 marks)
 - (e) Prove that $\underline{V}$ is a proper class. (2 marks)
2. This question assumes AC.
 - (a) Prove that, for every cardinal κ , there exists a singular cardinal κ' with $\kappa' > \kappa$. (6 marks)
 - (b) Suppose $\aleph_\alpha$ is a limit cardinal. Prove that there exists an increasing sequence of *initial ordinals* of length $\text{cf}(\aleph_\alpha)$ with limit $\aleph_\alpha$. (8 marks)
 - (c) Prove that, for every cardinal κ , there exists a singular cardinal κ' such that $\kappa' > \kappa$ and $\kappa' = \aleph_{\kappa'}$. (11 marks)

(Write your answer here.)

(More space for your answer to Question 4.)