Formal axiomatic set theory & foundations

(This lecture is non-examinable.)

We can reformulate our axioms as precise axioms in first-order logic.

Properties are expressed by formulas $\phi, \psi, \ldots$ constructed from variables $x, y, z, \ldots$ as below

- $x = y$
- $S(x)$ (x is a set)
- $x \in y$ (x is a member of the set y)
- $\phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi \mid \phi \leftrightarrow \psi \mid \neg \phi$
- $\forall x\, \phi \mid \exists x\, \phi$

Think of variables as ranging over the universe $\mathcal{U}$.

Our axioms can be expressed rather concisely as formulas. It is convenient to introduce some abbreviations

$$\forall x \in y \;\; \phi \quad \equiv \quad \forall x\,(x \in y \to \phi)$$

$$\exists x \in y \;\; \phi \quad \equiv \quad \exists x\,(x \in y \wedge \phi)$$

$$Ꙅ x \;\; \phi \quad \equiv \quad \exists y \;\; S(y) \wedge \forall x\,(\phi \leftrightarrow x \in y)$$

We can think of the last as introducing a new quantifier: "there are set-many x satisfying ϕ"

Formal axioms

(Memb) $\forall x \forall y \left(y \in x \to S(x) \right)$

(Extensionality) $S(x) \wedge S(y) \wedge \forall z (z \in x \leftrightarrow z \in y) \to x = y$

(Separation) $S(x) \to \text{Ƨ}y \, (y \in x \wedge \phi)$

(Pairing) $\text{Ƨ}z \, (z = x \vee z = y)$

(Powerset) $S(x) \to \text{Ƨ}y \left(S(y) \wedge \forall z \in y \;\; z \in x \right)$

(Indexed-union) $S(x) \wedge \forall y \in x \, \text{Ƨ}z \, \phi \to \text{Ƨ}z \, \exists y \in x \, \phi$

(Infinity) $\exists x \, (\exists y \in x \;\; S(y) \wedge \forall z \, \neg z \in y) \wedge$

$$\forall y \in x \, \exists z \in x \, \forall w \, (w \in z \leftrightarrow (w = y \vee w \in y))$$

Here the indexed-union axiom is equivalent to Union + Replacement from Lecture 1.

One benefit of having a formal axiomatisation is that first-order logic has <u>formal proof rules</u> which can be used to construct formally correct (and machine-checkable) proofs from the axioms.

Astonishingly, (almost) all the theorems of mathematics can be expressed as formulas in our first-order language, and formally! proved from our axioms (possibly with AC, or one of its variants, added).

Typically in mathematics the accepted set theory is Zermelo-Fraenkel set theory (ZF) extended by AC: abbreviated ZFC

ZF is:

Our axioms

+ $\forall x \; S(x)$ (everything is a set)

+ $\forall x \; \left(\exists y \; y \in x \right) \longrightarrow \exists y \in x \; \forall z \in x \;\; z \notin y)$

(the axiom of foundation: $\in$ is a well-founded relation)

The standard formulation does not have formulas $S(x)$ as they are unnecessary.

$$ZFC = ZF + AC$$

Another benefit of formalisation is that statements that principles of interest are <u>unprovable</u> become precise mathematical theorems.

We can show (by giving a mathematical proof) that a formula ϕ is <u>not derivable</u> from a set of formal axioms T.

We write $T \vdash \phi$ to say that ϕ is provable from the set of axioms T

We are interested also in showing

$T \nvdash \phi$ (ϕ is <u>not</u> provable from T)

or <u>equivalently</u>

$\mathrm{Con}(T + \neg\phi)$ (The theory $T + \neg\phi$ is consistent.)

Important examples of consistency results

$\mathrm{Con}(ZF) \Rightarrow \mathrm{Con}(ZFC)$ Gödel

$\mathrm{Con}(ZF) \Rightarrow \mathrm{Con}(ZF + \neg AC)$ Cohen

$\mathrm{Con}(ZF) \Rightarrow \mathrm{Con}(ZFC + GCH)$ Gödel

$\mathrm{Con}(ZF) \Rightarrow \mathrm{Con}(ZFC + \neg CH)$ Cohen

In brief:

AC is independent from ZF

(G)CH is independent from ZFC

Consistency theorems always require an assumption that some base theory is consistent. (In the examples above, the base theory is ZF.)

The assumption is needed because our axioms cannot prove $\mathrm{Con}(ZF)$ (this is due to Gödel's incompleteness theorem.)

In mainstream mathematical culture AC is accepted as a mathematical 'truth' and is thus adopted as an axiom

The consistency result for ZFC says that AC does not pose any consistency risks

Similarly the consistency results for CH and $\neg$CH show that neither poses any consistency risk

However, neither CH nor $\neg$CH is recognised as a mathematical 'truth' that is self-evident enough to warrant axiomatic status.

Some other independent statements do pose consistency risks. (WIC = $\exists$ weakly inaccessible cardinal, SIC = $\exists$ strongly " ")

$$\mathrm{Con}(ZF) \Rightarrow \mathrm{Con}(ZFC + \neg WIC)$$

$$\mathrm{Con}(\underbrace{ZF + WIC}_{\text{This cannot be weakened to } ZF.}) \Rightarrow \mathrm{Con}(ZFC + SIC)$$

It is possible that ZFC + SIC might be inconsistent even with ZF being consistent, however this can only happen if ZF + WIC is inconsistent.

<u>Moral</u> The consistency of an inaccessible cardinal requires a <u>leap of faith</u> beyond whatever faith we have in the consistency of ZF

Requiring a leap of faith is typical for large cardinal axioms.

Large cardinal axioms have two kinds of motivation

- Intrinsic – one sees them as making sense in their own right, perhaps even as expressing a deep 'truth' that incredibly large sets 'exist'.

- Extrinsic – they lead to pleasant (and useful?) mathematical consequences.

As an example of the second type:

<u>Theorem</u> (Martin, Steel)
Suppose there exist infinitely many Woodin cardinals.

a Woodin cardinal is a kind of large cardinal much larger than an inaccessible cardinal

Then Projective Determinacy (PD) holds.

PD: every projective subset of $\mathbb{N}^{\mathbb{N}}$ is determined.

(The projective sets extend the Borel sets.)

<u>Moreover</u> $\mathrm{Con}(ZFC + PD) \Rightarrow \mathrm{Con}(ZFC + IWC)$

IWC: $\exists$ infinitely many Woodin cardinals

PD has useful consequences

<u>Theorem</u> If PD holds then every projective set:

- has the perfect set property
- is universally measurable
- ...

<u>Theorem</u> (Woodin). If IWC is consistent then the theory:

$$ZF + AD$$

is consistent

The full <u>Axiom of Determinacy</u>: <u>every</u> subset of $\mathbb{N}^{\mathbb{N}}$ is determined

ZF + AD is a <u>very</u> interesting theory:

- AC is false
- CC is true
- Every subset of $\mathbb{R}$ is Lebesgue measurable
- Every subset of $\mathbb{R}$ has the perfect set property
- CH is true
- ACH is false

The big question: how can we be sure the axioms of set theory are 'true'?

Anyway what does 'truth' mean for:

- Properties of natural numbers? (E.g. $P \neq NP$)
- Properties of real/complex numbers? (E.g. Riemann hypothesis.)
- Properties of sets of real numbers? (CH)
- General set-theoretic statements? (AC)

and how do we know that the consequences of set-theoretic proofs are true?