



DISCRETE MATHEMATICS 2

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4. SYMMETRIES OF GRAPHS

The notes on this section are heavily based on a set of course notes written by Prof. Primož Šparl.

We shall assume some standard notation. A graph Γ consists of a set $V(\Gamma)$ of *vertices* and an edge-set $E(\Gamma)$ of (unordered) pairs of vertices. For two vertices $u, v \in V(\Gamma)$, we often denote $u \sim v$ instead of $\{u, v\} \in E(\Gamma)$. A *path* is a sequence of vertices $u_0, \dots, u_\ell$ such that $u_{i-1} \sim u_i$ for all $i \in \{1, \dots, \ell\}$. A path starting on u and finishing on v is a *uv-path*. A graph is connected if for every two vertices u, v there is a *uv-path*. The neighbourhood of a vertex v in Γ is the set

$$N(v) = \{u \in V(\Gamma) : u \sim v\},$$

and we say that v is of *valency* or *degree* d if $|N(v)| = d$.

We shall also assume some familiarity with certain standard families of graphs: We denote by $\mathcal{K}_m$ the complete graph on n vertices. The graph $\mathcal{K}_{m,n}$ is the complete bipartite graph with m vertices on one part and n vertices on the other. The cycle with n vertices is denoted $\mathcal{C}_n$.

For a graph Γ we denote by $\text{Aut}(\Gamma)$ its *automorphism group*, that is

$$\text{Aut}(\Gamma) = \{\alpha \in S_{V(\Gamma)} : v \sim u \text{ if and only if } \alpha u \sim \alpha v\}.$$

If Γ is a graph and $G \leq \text{Aut}(\Gamma)$ we say that Γ is *G-vertex-transitive* (*G-VT*) and *G-edge-transitive* (*G-ET*) if G acts transitively on $V(\Gamma)$ and $E(\Gamma)$, respectively. If $G = \text{Aut}(\Gamma)$ we just say that Γ is *vertex-transitive* (*VT*) or *edge-transitive* (*ET*).

In this notes we will focus mostly on vertex-transitive graphs but other classes of symmetry conditions are also interested.

A fairly obvious observation is the following.

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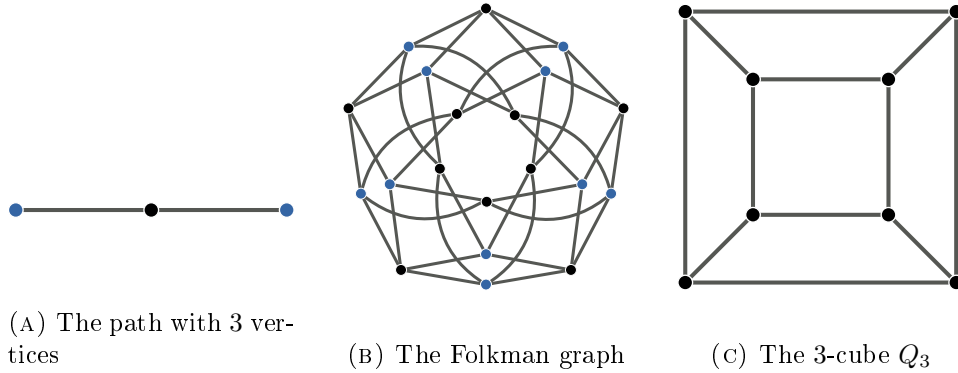


FIGURE 1

Remark 4.1. If Γ is a vertex-transitive graph, then every vertex v has the same valency. Such graphs are called *regular* and if the valency is k they are often called k -regular.

Proof. If α is an automorphism mapping v to u , then $\alpha|_{N(v)} : N(v) \rightarrow N(u)$ is a bijection that shows that $|N(v)| = |N(u)|$. $\square$

The previous result, although obvious impose an important restriction on vertex transitive graphs. It is not hard to find a small example of a non-vertex transitive graph. For example the path on 3 vertices (see Figure 1a) is not vertex transitive because it has two vertices of valency 1 and one vertex of valency 2. However, finding a small example of a regular graph that is not vertex-transitive is not trivial. The smallest example is the Folkman graph (see Figure 1b), which is a 4-regular graph with 20 vertices. This graph has two orbits on vertices. It is not hard to see that the blue vertices and the black vertices are in different orbits: every blue vertex has a (blue) vertex that have the exact same neighbourhood. This is not the case with the black vertices, hence they cannot be on the same orbit. It is also easy to see that any two vertices belong to the same orbit. A slightly more interesting (but not hard) task is to show that all the black vertices are in the same orbit.

There are infinite families of vertex transitive graphs: the complete graph K_n on n vertices, the cycle C_n on n vertices. The cube Q_3 (Figure 1c) is also vertex transitive.

An important family of vertex transitive graphs are the so called *Cayley graphs*. They were introduced as early as 1878 by Cayley as a graphic representation of a group. In fact, they were originally called *color group*.

Definition 4.2. Given a group G and a set $S \subseteq G$ such that

- $1_G \notin S$,
- $S^{-1} := \{s^{-1} : s \in S\} = S$. The *Cayley graph* (of G with respect to S) $\text{Cay}(G, S)$ is the graph whose vertex set is G and such that $x \sim y$ if and only if $y^{-1}x \in S$.

Observe that the fact that $1_G \notin S$ implies that $\text{Cay}(G, S)$ has no loops and since $S = S^{-1}$ then $y^{-1}x \in S$ if and only if $x^{-1}y \in S$. Therefore, $\text{Cay}(G, S)$ is actually a well-defined graph.

Example 4.3. There are natural examples of Cayley graphs:

- (a) The complete graph is a Cayley graph for every group of n elements. More precisely, $K_n \cong \text{Cay}(G, G \setminus \{1_G\})$.
- (b) The cycle C_n is isomorphic to $\text{Cay}(\mathbb{Z}_n, \{1, -1\})$ (recall that the trivial element in $\mathbb{Z}_n$ is 0).
- (c) The cube Q_3 is $\text{Cay}(\mathbb{Z}_2^3, \{e_1, e_2, e_3\})$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.
- (d) We can define the n -cube Q_n as the Cayley graph $\text{Cay}(\mathbb{Z}_2^n, \{e_1, \dots, e_n\})$ where $\{e_1, \dots, e_n\}$ is the standard basis for the vector space $\mathbb{Z}_2^n$.

Proposition 4.4. *Every vertex in $\text{Cay}(G, S)$ has valency $|S|$ and the graph $\text{Cay}(G, S)$ is connected if and only if $\langle S \rangle = G$.*

Proof. For a given vertex $x \in G$, the neighbours of x are precisely the elements of the form xs for $s \in S$. It follows that x has exactly $|S|$ neighbours.

Assume that $\text{Cay}(G, S)$ is connected and let x be any element of G . Since G is connected there is a path P from the vertex 1_G to x . Let $1_G = x_0, x_1, \dots, x_\ell = x$ the sequence of vertices induced by P . Observe that since $1_G \sim x_1$, then $x_1 \in S$, say $x_1 = s_1$. Similarly, $x_2 = x_1 s_2 = s_1 s_2$ for some $s_2 \in S$. In general, if $0 \leq i \leq \ell$ then $x_i = x_{i-1} s_i$ for some $s_i \in S$. It follows that $x = x_\ell = s_1 \cdots s_\ell \in S$; hence $\langle S \rangle = G$.

The other implication is similar. If $G = \langle S \rangle$, then every vertex x can be written as a product $s_1 \cdots s_\ell$ of elements in S . This induces a path from $1_G := x_0$ to $x := x_\ell$ with vertex sequence $x_0, \dots, x_\ell$ by defining $x_i = x_{i-1} s_i$. This implies that $\text{Cay}(G, S)$ is connected. $\square$

Now we turn our attention to symmetry properties of Cayley graphs.

Proposition 4.5. *The action of G on itself by left multiplication induces an action of G on $\text{Cay}(G, S)$ by automorphism. Since this action is transitive, every Cayley graph is vertex transitive.*

Proof. The left action of an element $g \in G$ is given $g : x \mapsto gx$ for $x \in G$. This induces a permutation of G , i.e. a permutation of the vertices of $\text{Cay}(G, S)$. We only need to show that this permutation maps edges to edges:

$$\begin{aligned}
 x \sim y &\iff y^{-1}x \in S \\
 &\iff y^{-1}g^{-1}gx \in S \\
 &\iff (gy)^{-1}(gx) \in S \\
 &\iff gx \sim gy.
 \end{aligned}$$

$\square$

The previous proposition shows that every Cayley graph is a vertex transitive graph. The converse is not true (see [Exercises 4.3](#) and [4.14](#)). In fact, the Petersen graph is the smallest example of a non-Cayley vertex-transitive graph.

Observe that the action of G on itself is not only transitive, but free, meaning that the only group element that fixes a point is 1_G . This implies that given any two vertices x, y in $\text{Cay}(G, S)$, there exists exactly one automorphism in G mapping x to y , namely, the automorphism induced by the group element yx^{-1} . It turns out that this property characterises Cayley graphs, as it is shown in the following theorem. This result is classical one on the theory of symmetries of graphs and it was first proved by G. Sabidussi in 1958.

Theorem 4.6 (Sabidussi, 1958). *A graph Γ is a Cayley graph if and only if its automorphism group has a subgroup G acting transitively and freely on the set of vertices of Γ .*

Proof. One implication was previously discussed. If $\Gamma = \text{Cay}(G, S)$, then G acts free and transitively.

Assume that for a graph Γ there exists a group $G \leq \text{Aut}(\Gamma)$ acting free and transitively on the vertices of Γ . Pick a fixed vertex $v_0 \in V(\Gamma)$. Consider the mapping $\varphi : x \mapsto g_x$ from the $V(\Gamma)$ to G where g_x is the *unique* element of G that maps the vertex v_0 to x . Observe that $\varphi(v_0) = 1_G$. Let $u_1, \dots, u_k$ be the neighbours of v_0 in Γ . Let $S = \{g_{u_1}, \dots, g_{u_k}\}$ and consider the graph $\text{Cay}(G, S)$. The function φ is a bijection between $V(\Gamma)$ and $G = V(\text{Cay}(G, S))$; let us show that it is a graph isomorphism.

Let x and y be two vertices in Γ such that $x \sim y$. We need to show that $\varphi(x) = g_x \sim g_y = \varphi(y)$. First, observe that the graph automorphism g_y^{-1} satisfies

$$g_y^{-1}(\{x, y\}) = \{g_y^{-1}(x), g_y^{-1}(y)\} = \{g_y^{-1}(x), v_0\},$$

which implies that $g_y^{-1}(x) = u_i$ for some $i \in \{1, \dots, k\}$.

Now consider the automorphism $g_y^{-1}g_x$

$$g_y^{-1}g_x(v_0) = g_y^{-1}(x) = u_i.$$

It follows that $g_y^{-1}g_x = g_{u_i} \in S$, that is $g_x \sim g_y$ in $\text{Cay}(G, S)$. This proves that φ is a graph isomorphism. $\square$

Proposition 4.7. *Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph of an abelian group G . If G is not an elementary abelian 2-group, that is, isomorphic to the direct product of many copies of $\mathbb{Z}_2$, then G is a proper subgroup of the automorphism group $\text{Aut}(\Gamma)$. Moreover, if the valence of Γ is at least 3, then the girth (the length smallest cycle of Γ) is at most 4.*

Proof. Consider the mapping $\alpha : G \rightarrow G$ given by $g \mapsto g^{-1}$. Since G is abelian we have that

$$g \sim h \iff h^{-1}g \in S \iff gh^{-1} \in S \iff g^{-1} \sim h^{-1} \iff \alpha(g) \sim \alpha(h).$$

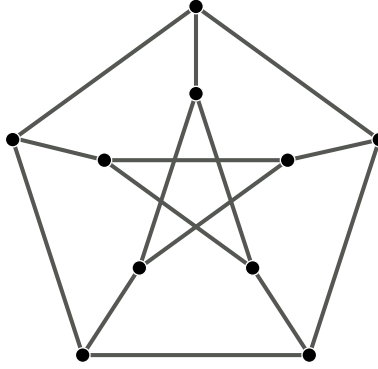


FIGURE 2. The Petersen graph

So clearly $\alpha \in \text{Aut}(\Gamma)$. The permutation α is a non-trivial element in $\text{Aut}(\Gamma)$ (since G is not elementary abelian 2-group) that fixes 1_G , hence cannot be induced by an element of G . If $|S| \geq 3$ we can pick two elements $a, b \in S$ such that $a \notin \{b^{-1}, b\}$. Then the path $(1_G, a, ab, b)$ is a 4-cycle in Γ . $\square$

The Petersen graph (see [Figure 2](#)) is a vertex transitive graph that is not Cayley (see [Exercise 4.14](#)). In fact it is the smallest non-Cayley vertex-transitive graph.

The problem of determining those numbers n for which there exists a vertex-transitive graph of order n which is not a Cayley graph was posed by Marušič in 1983. It has proved to be a highly non-trivial problem and a complete solution is, to the date, still unknown. The following proposition is a very basic basic result on the lines of this problem.

Proposition 4.8. *If Γ is a vertex-transitive graph on p vertices for prime number p , then Γ is a Cayley graph.*

Proof. Since Γ is vertex-transitive, then the Orbit-Stabiliser Theorem implies that the size p divides the size of $\text{Aut}(\Gamma)$. By Cauchy Theorem, there exists an element $\alpha \in \text{Aut}(\Gamma)$ of order p . In particular α is not trivial and if v is a vertex moved by α , then $|\langle \alpha \rangle x|$ divides p , hence $|\langle \alpha \rangle x| = p$. It follows that the group $\langle \alpha \rangle$ acts free and transitive on the vertices of Γ and by [Theorem 4.6](#), Γ is a Cayley graph. $\square$

Cayley graphs offer a way to build vertex transitive graphs, but unfortunately not every vertex transitive graph is a Cayley graph. We now explore a construction that generalises that of Cayley graphs and that will give us a way to describe every vertex transitive graph. Let us explore basic properties of vertex transitive graphs that will inspire the construction.

Proposition 4.9. *Let Γ be a vertex transitive graph. Let v denote a fixed vertex and denote by H the stabiliser of v under the automorphism group of Γ . Denote by S the set*

$$\{g \in \text{Aut}(\Gamma) : g(v) \in N(v)\}.$$

Then:

- (a) Every vertex u of Γ can be identified with the coset gH with $g \in \text{Aut}(\Gamma)$ an automorphism mapping u to v .
- (b) $S = \bigcup_{s \in S} HsH = \{h_1sh_2 : h_1, h_2 \in H \text{ and } s \in S\}$.
- (c) S is closed under inverses, that is $s \in S$ if and only if $s^{-1} \in S$ and $H \cap S = \emptyset$.
- (d) g_1v is adjacent to g_2v in Γ if and only if $g_2^{-1}g_1 \in S$.
- (e) Γ is connected if and only if $\langle H, S \rangle = G$.

Proof.

- (a) It follows directly from the fact that Γ is vertex-transitive: the action of $\text{Aut}(\Gamma)$ on vertices is equivalent to the action of $\text{Aut}(\Gamma)$ on the left cosets of H .
- (b) Clearly $S \subseteq \bigcup_{s \in S} HsH$. Take $h_1, h_2 \in H$ and $s \in S$. then $h_1sh_2v = h_1sv = h_1u$ for some $u \in N(v)$. Since $h_1 \in H$, then h_1 permutes the elements of $N(v)$ then $h_1u \in N(v)$ hence $h_1sh_2v \in N(v)$, which implies $h_1sh_2 \in S$.
- (c) Take $s \in S$, meaning $sv \in N(v)$. Then $\{v, sv\}$ is an edge of Γ . Since $s^{-1} \in \text{Aut}(\Gamma)$, then

$$s^{-1}\{v, sv\} = \{s^{-1}v, v\}$$

is an edge in Γ , which implies that $s^{-1}v \in N(v)$, or equivalently, $s^{-1} \in S$. Every element in $h \in H$ fixes v , in particular $hv \notin N(v)$, hence $h \notin S$.

- (d) $\{g_1v, g_2v\}$ is an edge if and only if $\{g_2^{-1}g_1v, v\}$ is a edge, and this occurs if and only if $g_2^{-1}g_1 \in S$.
- (e) This follows the same ideas as the proof of [Proposition 4.4](#).

□

Proposition above inspires a construction that we shall call *Sabidussi coset graphs* or simply *coset graphs*. This construction was introduced by Sabidussi in 1964. We should remark that the term coset graph is sometimes used to name *Shereier coset graphs* which are a different construction. Sabidussi coset graphs are also called *Generalised orbital graphs*.

Definition 4.10. Given a group G , a subgroup H and a set $S \subseteq G$ that satisfy:

- $S = \bigcup_{s \in S} HsH$,
- $S^{-1} = S$,
- $S \cap H = \emptyset$.

Then the *Sabidussi coset graph* $\text{SabCos}(G, H, S)$ is the graph whose vertex set are the left cosets of H in G and whose edges are given by

$$xH \sim yH \iff y^{-1}x \in S.$$

First observe that this relation actually defines a graph. Assume that x_1, x_2, y_1, y_2 are elements in G such that $x_1H = x_2H$, $y_1H = y_2H$ and

$y_1^{-1}x_1 \in S$. Since $x_1H = x_2H$, then $x_1h_1 = x_2$ for some $h_1 \in H$. Similarly $y_1h_2 = y_2$ for some $h_2 \in H$. This implies that

$$y_2^{-1}x_2 = (y_1h_2)^{-1}(x_1h_1) = h_2^{-1}y_1^{-1}x_1h_1 \in S,$$

where the last equality follows from the facts that $y_1^{-1}x_1 \in S$ and S is a union of double cosets of H . This shows that the defining relations for the edges of $\text{SabCos}(G, H, S)$ is well defined on cosets. The relation is symmetric because $S = S^{-1}$ and the graph has no loops because $H \cap S$.

Proposition 4.11. *Let G be a group, $H \leq G$ and $S \subseteq G$ satisfying the condition in Definition 4.10. Let $\Gamma = \text{SabCos}(G, H, S)$. Then the action of G by left multiplication on left cosets of H induces an action of G on Γ by automorphisms. The group G acts transitively on vertices.*

Proof. The action of left multiplication on left cosets of H is transitive. We just need to prove that every element in G actually induces automorphisms of Γ . Observe that

$$xH \sim yH \iff y_{-1}x \in S \iff y^{-1}g^{-1}gx \in G \iff gyH \sim gxH.$$

This prove that G actually induces automorphisms of Γ . $\square$

We have prove that every coset graph is vertex-transitive. The converse is also true and therefore we have a characterisation of vertex transitive graphs in terms of groups.

Theorem 4.12. *Every vertex transitive graph Γ is isomorphic to a Sabidussi coset graph.*

Proof. We actually have mostly proved this theorem. Let $G = \text{Aut}(\Gamma)$, v a vertex of Γ and $H = \text{Stab}_G(v)$. Define $S = \{s \in G : sv \in N(v)\}$. By part (a) of Proposition 4.8 the function $\varphi : V(\Gamma) \rightarrow G/H$ given by $\varphi(u)$ Moreover, part (d) of the same result implies that φ is a graph isomorphism. $\square$

We finish this section with an example in Figure 3 we show how to build the Petersen graph as a coset graph. The group G is the alternating group A_5 and the group $H = \langle (1\ 2\ 3), (1\ 2)(4\ 5) \rangle$. A simple calculation shows that $S = (2\ 4)(3\ 5)H \cup (1\ 4)(2\ 5)H \cup (1\ 4)(3\ 5)H$. In the figure we only wrote the representatives of the coset of each vertex.

The example above show that we do not need to use the full automorphism group of the vertex transitive graph to build as a coset graph. The construction works perfectly if we use any vertex-transitive subgroup of $\text{Aut}(\Gamma)$. For the Petersen graph, the automorphism group is S_5 , while we build it using the alternating group.

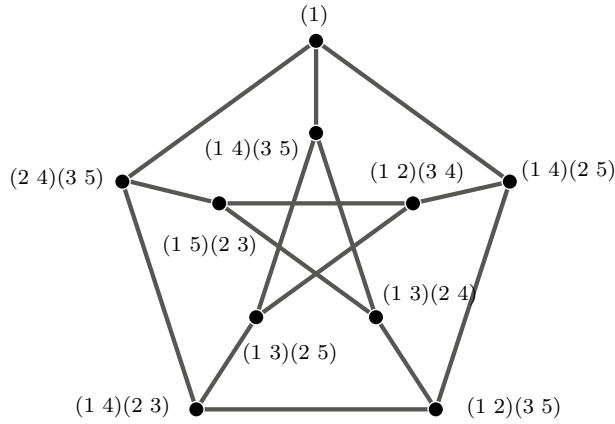


FIGURE 3. The Petersen graph as a coset graph

Exercises.

- 4.1 Prove that there exists a graph on n vertices, all of them of valency k if and only if $n \geq k + 1$ and nk is even.
- 4.2 Find the automorphism group of the cycle $\mathcal{C}_n$.
- 4.3 Consider the graph Γ whose vertices are the sets of size 2 of $\{1, 2, 3, 4, 5\}$ and so that $A \sim B$ if and only if $A \cap B = \emptyset$.
 (a) Prove that Γ is isomorphic to the Petersen graph.
 (b) Find the automorphism group of Γ .
- 4.4 Let H and K two subgroups of a group G . Assume that
- $H \triangleleft G$ and $K \triangleleft G$,
 - $G = \langle H, K \rangle$,
 - $H \cap K = \{1\}$.
- Prove that $G \cong H \times K$.
- 4.5 Let H and K two groups and $\theta : K \rightarrow \text{Aut}(H)$ and for $k \in K$ denote by θ_k the mapping $\theta(k) : H \rightarrow H$. Consider the set $H \times K$ and the operation $*_\theta$ given by $(h_1, k_1) *_\theta (h_2, k_2) = (h_1 \theta_{k_1}(h_2), k_1 k_2)$. Prove that the pair $(H \times K, *_\theta)$ is a group. This group is called the *(external) semidirect product of H and K with respect to θ* and is often denoted $H \rtimes_\theta K$.
- 4.6 Let H and K two subgroups of a group G . Assume that
- $H \triangleleft G$,
 - $G = \langle H, K \rangle$,
 - $H \cap K = \{1\}$.
- Prove that $G \cong H \rtimes_\theta K$, where $\theta_k(h) = khk^{-1}$. This is often called the *(internal) semidirect product of H and K* and it is usually denoted by $H \rtimes K$.
- 4.7 Prove that a complete bipartite graph $\mathcal{K}_{m,n}$ is vertex transitive if and only if $m = n$.

- 4.8 Find the automorphism group of the complete bipartite graph $\mathcal{K}_{m,n}$.
Hint: consider the cases $n \neq m$ and $n = m$ separately.

- 4.9 Let Q_n denote the n -cube graph. Prove that

$$\text{Aut}(Q_n) \cong \mathbb{Z}_2^n \rtimes_{\theta} S_n,$$

where $\theta(\sigma) : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^n$ is defined by

$$\theta(\sigma) : (v_1, \dots, v_n) = (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(n)}),$$

for every vector $(v_1, \dots, v_n) \in \mathbb{Z}_2^n$. That is, θ is the natural action of S_n permuting the coordinates of $\mathbb{Z}_2^n$.

- 4.10 Assume that a graph Γ has m_1 connected components isomorphic to Δ_1 , m_2 connected components isomorphic to Δ_2 , etc. Such that $\Delta_i \cong \Delta_j$ if and only if $i = j$. Prove that the automorphism group of Γ is isomorphic to

$$(\text{Aut}(\Delta_1)^{m_1} \times \dots \times \text{Aut}(\Delta_\ell)^{m_\ell}) \rtimes (S_{m_1} \times \dots \times S_{m_\ell}).$$

- 4.11 The *cartesian product* of two graphs Γ and Δ is denoted by $\Gamma \square \Delta$ and defined by $V(\Gamma \square \Delta) = V(\Gamma) \times V(\Delta)$ and the edges of $\Gamma \square \Delta$ are given by

$$(u_1, v_1) \sim (u_2, v_2) \Leftrightarrow \begin{cases} u_1 = u_2 \text{ and } v_1 \sim v_2 & \text{or} \\ v_1 = v_2 \text{ and } u_1 \sim u_2 \end{cases}$$

- (i) Prove that if Γ and Δ are vertex transitive, then so is $\Gamma \square \Delta$.
(ii) Prove that $\text{Aut}(\Gamma) \times \text{Aut}(\Delta) \leq \text{Aut}(\Gamma \square \Delta)$.
(iii) Give an example of two graphs Γ and Δ such that $\text{Aut}(\Gamma) \times \text{Aut}(\Delta)$ is a proper subgroup of $\text{Aut}(\Gamma \square \Delta)$.
- 4.12 The *lexicographic product* of two graphs Γ and Δ is denoted by $\Gamma[\Delta]$ and defined by $V(\Gamma[\Delta]) = V(\Gamma) \times V(\Delta)$ and the edges of $\Gamma[\Delta]$ are given by

$$(u_1, v_1) \sim (u_2, v_2) \Leftrightarrow \begin{cases} u_1 \sim u_2 \text{ in } \Gamma & \text{or} \\ u_1 = u_2 \text{ and } v_1 \sim v_2 \text{ in } \Delta \end{cases}$$

Assume that Γ has m vertices. Observe that the action of $\text{Aut}(\Gamma)$ on $V(\Gamma)$ induces an action θ of $\text{Aut}(\Gamma)$ on $(\text{Aut}(\Delta))^m$.

- (i) Prove that if Γ and Δ are vertex transitive, then so is $\Gamma[\Delta]$.
(ii) Prove that the group

$$\text{Aut}(\Delta)^m \rtimes_{\theta} \text{Aut}(\Gamma)$$

is a subgroup of $\text{Aut}(\Gamma[\Delta])$.

- (iii) Give an example of two graphs Γ and Δ such that $\text{Aut}(\Gamma)^m \rtimes_{\theta} \text{Aut}(\Delta)$ is a proper subgroup of $\text{Aut}(\Gamma[\Delta])$.
- 4.13 Prove that a graph is a Cayley graph if and only if its complement is a Cayley graph.

- 4.14 Prove that if p is an odd prime then there are only two groups of order $2p$, namely $\mathbb{Z}_{2p}$ and D_p . Hint: prove that $\mathbb{Z}_{2p} \cong \mathbb{Z}_p \times \mathbb{Z}_2$ and $D_p \cong \mathbb{Z}_p \rtimes \mathbb{Z}_2$. Use this to prove that the Petersen graph is not a Cayley graph.
- 4.15 Prove that there is no vertex-transitive graph of order at most 9 which is not a Cayley graph.
- 4.16 Prove that the cube graph is isomorphic to some generalised orbital graph of the symmetric group S_4 .