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DISCRETE MATHEMATICS 2

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5. Ramsey theory

In this section we explore what is called *Ramsey theory*. The basic idea is given in the following example:

Example 5.1. Six people are waiting in the lobby of a hotel. Prove that there are either three of them who know each other, or three of them who do not know each other.

Solution. We can model the problem with a coloured complete graph. Let us use one vertex for each person and colour the edge between two of them with red if they know each other and with blue if they dont. Pick a vertex v, there must be a colour so that at least three of the edges incident to v. With out loss of generality, we may assume that this colour is red and consider the set B of the the endpoints of the red edges of incident to v. If between any two elements in B there is a red edge, then we can use v to complete a red triangle (see Figure 1a), which will give us a set of three people who know each other. Otherwise, any edge in betweeen any two elements in B is blue and by construction $|B| \geqslant 3$. It follows that we can pick three elements in B that form a blue triangle (see Figure 1b). That gives us a set of three people that do not know each other. $\square$

The previous example is the first example of the *Ramsey theory*, which is named after Frank Ramsey who was a British mathematician that was mostly interested on philosophy and logic. He died at the age of 26 in 1930, the same year his paper *On a problem of formal logic* was published. This paper was the formal beginning of what we know as Ramsey theory.

Example 5.1 is one of the first instances of a more general problem.

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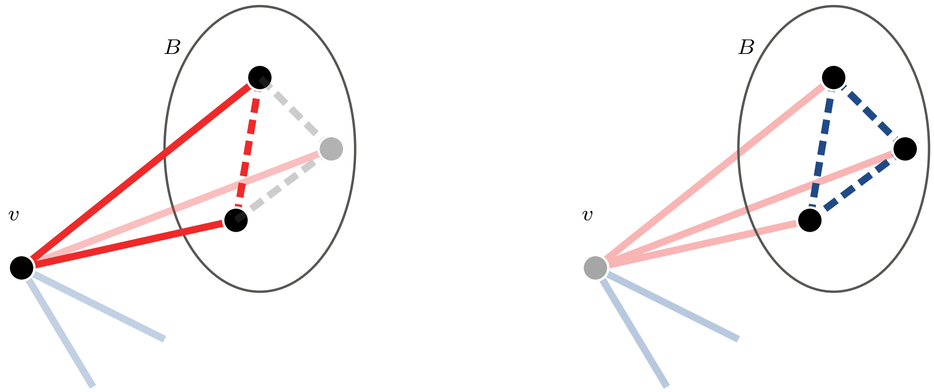

(A) (B)

FIGURE 1. $R(3,3) \leqslant 6$

A colouring of the edges of $\mathcal{K}_n$ with s colours is called a *s-colouring*. If $V_k \subseteq V(\mathcal{K}_n)$ is a set with k vertices such that every edge between vertices of V_k is red (say) we say that the s-colouring *contains* a red $\mathcal{K}_k$.

Definition 5.2. Given positive integers k and ℓ, we say that a number $n \in \mathbb{N}$ satisfies the *Ramsey condition for k and ℓ* if for every 2-colouring of the complete graph $\mathcal{K}_n$ with colours red and blue (say), contains a red $\mathcal{K}_k$ or a blue $\mathcal{K}_\ell$.

Clearly, if n satisfies the Ramsey condition for k and ℓ, then any other number $m \geqslant n$ also satisfies the Ramsey condition for k and ℓ. We denote by $R(k,\ell)$ the smallest of such numbers. In principle, this number $R(k,\ell)$ may not exist for arbitrary k and ℓ, we shall prove in Theorem 5.3 that this number actually exists.

Example 5.1 shows that $R(3,3) \leqslant 6$. The colouring of $\mathcal{K}_5$ in Figure 2 proves that $R(3,3) > 5$, which implies that $R(3,3) = 6$.

Before proving the general theorem let us explore the small cases. Every 2-colouring of the edges of K_n has an *opposite* colouring given by swapping the colours. This implies that $R(k,\ell) = R(\ell,k)$. If $k = 1$ or $\ell = 1$, then the problem becomes trivial and $R(1,k) = 1 = R(1,k)$ for any k. Less trivially, we can see that $R(k,2) = k$. If we colour the edges of $\mathcal{K}_k$ with two colours, red and blue and at least one edge is blue, then the colouring contains a blue $\mathcal{K}_2$. Otherwise the whole $\mathcal{K}_k$ is red.

Let us prove what is often called the Ramsey theorem for graphs.

Theorem 5.3. *Let $k, \ell \geqslant 2$, then the number $R(k,\ell)$ exists and satisfies*

$$R(k,\ell) \leqslant R(k-1,\ell) + R(k,\ell-1).$$

Proof. We proceed by induction over $k + l$. The smallest possible value of $k = 2 = \ell$ then the theorem follows from the previous discussion. Assume

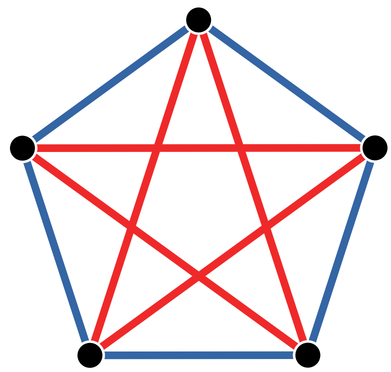

FIGURE 2. $R(3,3) > 5$

inductively that both $R(k, \ell-1)$ and $R(k-1, \ell)$ exist. Take a 2-colouring of the complete graph on $n = R(k-1,\ell) + R(k, \ell-1)$ vertices with colours red and blue. We will prove that this colouring contains either a red $\mathcal{K}_k$ or a blue $\mathcal{K}_\ell$ Pick a vertex v and let r and b be the number of red and blue edges incident to v, respectively. If both $r \leqslant R(k-1,\ell) - 1$ and $b \leqslant R(k, \ell-1)$ hold, then the number of edges incident to v are

$$R(k-1,\ell) + R(k,\ell-1) - 1 = r + b \leqslant R(k-1,\ell) + R(k,\ell-1) - 2,$$

which is obviously a contradiction. It follows that either $r \geqslant R(k-1,\ell)$ or $b \geqslant R(k, \ell-1)$.

Assume that $r \geqslant R(k-1,\ell)$ and let B denote the endpoints of the r red edges incident to v. Since $r = |B| \geqslant R(k-1,\ell)$, then the induced 2-colouring of the complete graph $\mathcal{K}_r$ contains either a red $\mathcal{K}_{k-1}$ or a blue $\mathcal{K}_\ell$. In the former case we can complete this graph to a red $\mathcal{K}_k$ by attaching v while in the latter the blue $\mathcal{K}_\ell$ is contained in the colouring of the original $\mathcal{K}_n$.

The case $b \geqslant R(k, \ell-1)$ is analogous. □

The previous theorem gives us an upper bound on $R(k,\ell)$. In general this bound is loose, as we shall see in the following example.

Example 5.4. $R(3,4) = 9$.

Solution. From the bound in Theorem 5.3 we know that

$$R(3,4) \leqslant R(2,4) + R(3,3) = 4 + 6 = 10.$$

Let us show that every colouring of $\mathcal{K}_9$ contains either a red triangle or a blue $\mathcal{K}_4$. If any vertex v has 4 red edges the the fact that $R(2,4) = 4$ implies that the induced colouring on the graph determined by the endpoints of the red edges has a red $\mathcal{K}_2$ or a blue $\mathcal{K}_4$. In the former case, we can complete the red $\mathcal{K}_2$ to a red $\mathcal{K}_3$ by attaching v. In the latter we obtain directly a blue $\mathcal{K}_4$. With a similar idea we can show that if any vertex has at least 6 blue edges we can use the fact that $R(3,3) = 6$ to find a red triangle or a blue triangle that can be completed to a blue $\mathcal{K}_4$.

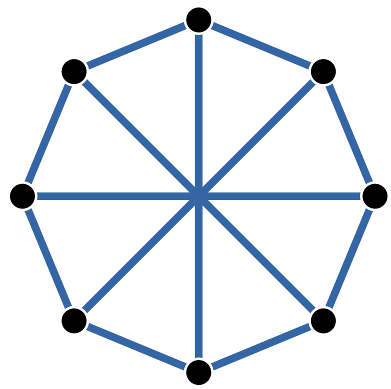

FIGURE 3. $R(3,4) > 8$

$k\backslash\ell$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		2	3	4	5	6	7	8	9
3			6	9	14	18	23	28	36
4				18	25				

TABLE 1. Known values for $R(k,\ell)$

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Therefore, we can assume that every vertex has at most 3 red edges and at most 5 blue edges. Since ever vertex has degree 8, it follows that every vertex has *exactly* 3 red edges and 5 blue edges. However, if we look at the graph induced by the red edges, this is a 3-valent graph with 9 vertices. This graph must have $\frac{3\times 9}{2}$ edges, which is clearly impossible.

We have only proved that $R(3,4) \leqslant 9$. The colouring in Figure 3 proves that $R(3,4) > 8$. In this figure we just only drew the blue edges. $\square$

Computing the exact value for $R(k,\ell)$ is not easy. In Table 1 we show the known values of the number $R(k,\ell)$ to the date of this notes. We only show the values that are known exactly but we must point out that some other values have be bounded, for example, it is known that $40 \leqslant R(3,10) \leqslant 42$.

The following result gives us another upper bound.

Proposition 5.5. *For every $k,\ell \geqslant 2$*

$$R(k,\ell) \leqslant \binom{k+\ell-2}{k-1}.$$

Proof. We can proceed by induction on $k+\ell$. If $k = 2 = \ell$ then

$$R(2,2) = 2 \leqslant \binom{2}{1} = 2.$$

By Theorem 5.3 and an inductive argument we have that

$$R(k,\ell) \leqslant R(k-1,\ell)+R(k,\ell-1) \leqslant \binom{k+\ell-3}{k-2}+\binom{k+\ell-3}{k-1} = \binom{k+\ell-2}{k-1},$$

where the last equality follows from Pascal formula. □

Corollary 5.6. *For every $k \geqslant 2$,*

$$R(k,k) \leqslant 4^{k-1}.$$

Proof. From Proposition 5.5

$$R(k,k) \leqslant \binom{2k-2}{k-1} \leqslant 2^{2k-2} = 4^{k-1}$$

□

We can also give upper bounds on the Ramsey numbers, an easy one is the following.

Proposition 5.7. *For every $k, \ell \geqslant 2$*

$$R(k,\ell) > (k-1)(\ell-1).$$

Proof. We need to exhibit a 2-colouring of the edges of $\mathcal{K}_n$ for $n = (k-1)(\ell-1)$ without red $\mathcal{K}_k$ or blue $\mathcal{K}_\ell$. Arrange the vertices n vertices on a grid with $k-1$ rows each containing $\ell-1$ points. Draw a blue edge in between any of them if and only if the belong in the same row. Given any k vertices, by the pigeonhole principle, two of them must be on the same row, hence the edge between them is blue. This implies that there is no red $\mathcal{K}_k$. If we remove all the red edges the connected components are the blue $\mathcal{K}_{\ell-1}$ induced by the vertices on a given row. This implies that there are no blue $\mathcal{K}_\ell$ either. □

The bound given by the proposition above is in general very loose. For example for $k = \ell = 3$ we obtain that $R(3,3) > 4$.

The bound in Corollary 5.6 shows that the numbers $R(k,k)$ grow at most exponentially with k it can be easily show that the actually grow exponentially. More precisely.

Proposition 5.8. *If $n < 2^{\frac{k}{2}}$ then there exists a 2-colouring of the edges of $\mathcal{K}_n$ without monochromatic $\mathcal{K}_k$. In particular $R(k,k) > 2^{\frac{k}{2}}$.*

Proof. We just explain the idea of the proof. There are a total of $2^{\binom{n}{2}}$ 2-colourings of the complete graph $\mathcal{K}_n$. The number of colourings with a monochromatic $\mathcal{K}_k$ on a given k-subset of vertices is $2\left(2^{\binom{n}{2}-\binom{k}{2}}\right)$. There are at most $\binom{n}{k}$ of such colourings, since a given colouring can be counted several times (think of the colouring where all the edges are red, for example). Using the fact that $\binom{n}{k} < \frac{n^k}{k!}$ and that $n < 2^{\frac{k}{2}}$ we can show that

$$2^{\binom{n}{2}} > 2^{\binom{n}{2}-\binom{k}{2}+1},$$

which proves our claim. □

A first generalisation of Theorem 5.3 is to admit colourings of many colours.

Definition 5.9. Given $k_1, \ldots, k_s$ we say that a number n satisfies the Ramsey condition for $k_1, \ldots, k_s$ if every s-colouring of the edges of the complete graph $\mathcal{K}_n$ has a complete $\mathcal{K}_{k_i}$ of colour i. We denote by $R(k_1, \ldots, k_s)$ the minimum number n that satisfies the Ramsey condition for $k_1, \ldots, k_s$.

As before, we should prove that this number exists.

Theorem 5.10. *Let $k_1, \ldots, k_s \geqslant 2$, then the number $R(k_1, \ldots, k_s)$ exists and satisfies*

$$R(k_1 \ldots, k_s) \leqslant R\left(k_i, R(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_s)\right)$$

for every $i \in \{1, \ldots, s\}$

Proof. We just prove the theorem for $i = 1$, the other inequalities follow in the exact same way. We proceed by induction over s. If $s = 2$, then the statement is precisely Theorem 5.3 by admitting the trivial convention that $R(\ell) = \ell$ for every $\ell \geqslant 2$. Assume that the number $R(\ell_1, \ldots, \ell_{s-1})$ exists for any numbers $\ell_1, \ldots, \ell_{s-1}$. Take an s-colouring of the edges of $\mathcal{K}_n$ with $n = R\left(k_1, R(k_2, \ldots, k_s)\right)$. Observe that this induces a 2-colouring of edges of $\mathcal{K}_n$ if we define that an edge is red if it was of colour 1 on the original colouring and it is blue otherwise.

Theorem 5.3 implies that there is either a red $\mathcal{K}_{k_1}$ which induces a $\mathcal{K}_{k_1}$ of colour 1 in the original colouring or there is a blue $\mathcal{K}_\ell$ for $\ell = R(k_2, \ldots, k_s)$. In this case, the blue $\mathcal{K}_\ell$ comes from complete graph $\mathcal{K}_\ell$ that has no edges of colour 1 in the original colouring. Our inductive hypothesis implies that this colouring of $\mathcal{K}_\ell$ has a $\mathcal{K}_{k_j}$ of colour j for some $j \geqslant 2$, as desired. $\square$

Colouring the edges of the complete graph is equivalent to colouring the 2-subsets of a set of n elements. A fairly general version of the Ramsey theorem uses colouring of the r-subsets of a set with n elements. We will just give the statement of the theorem without a proof. However we should point out that the prove for two colours (that is $s = 2$) is not conceptually more difficult than that for $r = 2$ (that is, colouring edges of a complete graph). Then the result can be generalised to many colours using exactly the same idea that the one used to generalise Theorem 5.3 to Theorem 5.10.

Theorem 5.11. *Gien $r \geqslant 2$, $s \geqslant 2$ and $k_1, \ldots, k_s \geqslant 2$ there exists a minimum number $n = R_r(k_1, \ldots, k_s)$ such that every if V is a set with $m \geqslant n$ elements, then every s-colouring of the r-subsets of V satisfies that there is $i \in \{1, \ldots, s\}$, a set $A \subseteq V$ with $|A| = k_i$ such that every r-subset of A is of colour i.*

Exercises.

5.1 18 teams participate at a round-robin soccer tournament. Prove that after eight rounds are played, we can still find three teams no two of which have played each other yet.

5.2 Each point of the space is colored either red or blue. Prove that either there is a unit square whose vertices are all blue, or there is a unit square that has at least three red vertices.

5.3 Prove that every edge-colouring of $\mathcal{K}_6$ with 2 colours has at least 2 monochromatic triangles (not necessarily of the same colour). Give a colouring of $\mathcal{K}_6$ with exactly two monochromatic triangles.

5.4 Prove that there exists a natural number $R(k_1, \ldots, k_s)$ such that if $n \geqslant R(k_1, \ldots, k_s)$ and we colour the edges of $\mathcal{K}_n$ with s colours, then there exists $i \in \{1, \ldots, s\}$ and a set V of k_i vertices that satisfis that all the edges of the complete graph induced by V are of colour i.

5.5 Prove that

$$R(\underbrace{3, \ldots, 3}_{s \text{ times}}) \leqslant 1 + s! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{s!}\right)$$

5.6 Let $n \geqslant 2$. Prove that $R(n+2, 3) > 3n$.

5.7 Prove that $R(3, 5) = 14$.

5.8 Prove that $R(4, 4) = 18$.

5.9 We colored each point of the space either red, or blue, or green or yellow. Prove that there is a segment of unit length with monochromatic vertices.

5.10 Prove that it is possible to colour each point of the plane either red or blue so that there is no regular triangle with sides of unit length and monochromatic vertices.

5.11 We coloured each point of the plane either red or blue. Let T be any right-angled triangle. Prove that there is a triangle that is congruent to T and has monochromatic vertices.

5.12 A company has 2002 employees, from 6 different countries. Each employee has a company identification card (ID) with a number from 1 to 2002. Prove that there is either an employee whose ID number is equal to the sum of the ID numbers of two of his compatriots, or there is an employee whose ID number is twice that on one of compatriots.

5.13 Let us colour each positive integer by one of the colors $c_1, \ldots, c_k$.

- Prove that there exists an integer $N(k)$ so that if $n > N$, then there are three integer a, b, c that are less than n, are of the same colour and satisfy $a + b = c$ ($a = b$ is allowed).
- Determine $N(2)$.
- Prove that $N(3) > 13$.

5.14 There are 9 participants in a convention. None of the participants speak more than 3 languages. It is also true that 2 of each 3 participants speak a common language. Show that there are 3 participants that speak a common language.