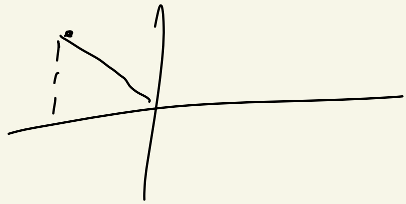
Matematika 1

2. izpit – rešitve

NALOGA 1



$z = -\frac{\sqrt{3}}{2} + \frac{1}{2} i$

V polarnem zapisu je

$r^2 = \frac{3}{4} + \frac{1}{4} = 1 \Rightarrow r = 1$

$\tan\varphi = \frac{|y|}{|x|} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\Rightarrow \varphi = \frac{\pi}{6} \Rightarrow \arg(z) = \frac{5\pi}{6}$

$\Rightarrow z = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$ 7

$w = z^{21} = \cos\frac{105\pi}{6} + i\sin\frac{105\pi}{6}$

ker je $105 = 17 \cdot 6 + 3$

$\Rightarrow w = \cos\left(17\pi + \frac{\pi}{2}\right) + i\sin\left(17\pi + \frac{\pi}{2}\right)$

$= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = \boxed{-i}$

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piszmy $z = x + iy$

Wtedy że $(x+iy)(x-iy-1) = -i+3$

$x^2+y^2-x-iy = -i+3$

$\Rightarrow x^2+y^2-x=3$ i $\underline{y=1}$ 6

$x^2-x-2=0$

$(x-2)(x+1)=0$

$\underline{x_1=2}, \underline{x_2=-1}$ 5

$\Rightarrow \boxed{z_1 = 2+i}, \boxed{z_2 = -1+i}$

NALOGA 2

(a) kvocientni kriterij:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\dfrac{x^{n+1}-x^{n}}{x^{2n+2}}\right|}{\left|\dfrac{x^{n}-x^{n-1}}{x^{2n}}\right|}$$

$$= \lim_{n\to\infty} \frac{|x|\,|\cancel{x^{n}-x^{n-1}}|}{x^{2}\,|\cancel{x^{n}-x^{n-1}}|} = \frac{1}{|x|} = L$$

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če je $L > 1$ => vrsta divergira

če je $L < 1$ => vrsta konvergira absolutno (torej konvergira)

$$\Rightarrow \frac{1}{|x|} < 1 \Rightarrow \boxed{|x| > 1}$$

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Preverimo, še x, za katere je $\frac{1}{|x|} = 1$

tj. $x = \pm 1$

$$x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1-1}{1} = \sum_{n=1}^{\infty} 0 = 0$$

konvergira

$$x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n - (-1)^{n-1}}{1}$$

3

$$= \sum_{n=1}^{\infty} (-1)^n (1 - (-1)) = 2 \sum_{n=1}^{\infty} (-1)^n$$

divergira

$\Rightarrow$ vrsta konvergira za $\boxed{x \in (-\infty, -1) \cup [1, \infty)}$

b) delna vsota

$$\sum_{n=1}^{N} \left(\frac{1}{x^n} - \frac{1}{x^{n+1}} \right)$$

$$= \frac{1}{x} - \cancel{\frac{1}{x^2}} + \cancel{\frac{1}{x^2}} - \cancel{\frac{1}{x^3}} + \dots - \frac{1}{x^{N+1}}$$

$$= \frac{1}{x} - \frac{1}{x^{N+1}}$$ 6

(c) vsota je ko $N \to \infty$

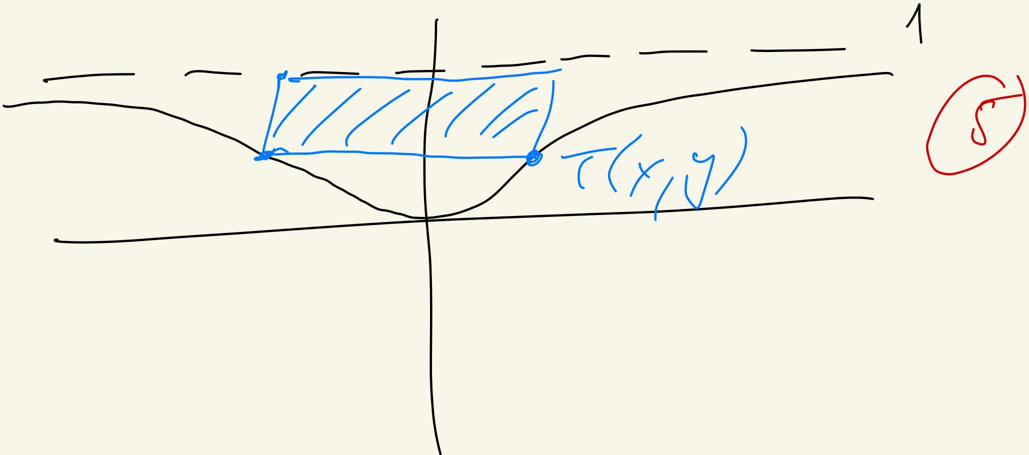
$$\lim_{N \to \infty} \left(\frac{1}{x} - \frac{1}{x^{N+1}} \right)$$ 6

$$= \boxed{\begin{cases} 0 & ; x = 1 \\ \frac{1}{x} & ; |x| > 1 \end{cases}}$$

ker $\lim_{N \to \infty} \frac{1}{x^{N+1}} = 0$ za $|x| > 1$

NALOGA 3

$$y = \frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = 1 - \frac{1}{1+x^2}$$



$$\text{ploščina} = S(x) = \text{dolžina} \cdot \text{širina}$$

$$= 2x \cdot (1-y)$$

$$= 2x\left(1 - 1 + \frac{1}{1+x^2}\right) = \frac{2x}{1+x^2} \quad S$$

iščemo ekstrem funkcije S

$$S'(x) = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

$\Rightarrow s'(x)=0 \iff 1-x^2=0$

$\Rightarrow x=\pm 1$

$\Rightarrow \boxed{x=1}$

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Preverimo, da gre za maksimum

$$s''(x) = \frac{-4x(1+x^2)^2-(2-2x^2)2(1+x^2)2x}{(1+x^2)^4}$$

$$s''(1) = \frac{-4\cdot 2^2}{2^4} < 0 \Rightarrow x=1$$

je maksimum

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$\Rightarrow$ maksimalna ploščina je

$$s(1) = \frac{2\cdot 1}{1+1} = \boxed{1}$$

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NALOGA 9

$$V = \pi \int_0^\infty f(x)^2\,dx = \pi \underbrace{\int_0^\infty e^{-2x}\sin^2 x\,dx}_{I}$$

PER PARTES

$$u = \sin^2 x$$

$$du = 2\sin x\cos x = \sin(2x)$$

$$dv = e^{-2x}\,dx$$

$$v = -\frac{1}{2}e^{-2x}$$

$$\Rightarrow I = -\frac{1}{2}e^{-2x}\sin^2 x\Big|_0^\infty$$

limita je enaka 0, ker je $\sin^2 x$ omejena

$$+ \int_0^\infty \frac{1}{2}e^{-2x}\sin(2x)\,dx$$

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$$= \frac{1}{2}\int_0^\infty e^{-2x}\sin(2x)\,dx$$

PER PARTES

$u = \sin(2x)$

$du = \cos(2x) \cdot 2$

$dv = e^{-2x} dx$

$v = -\frac{1}{2} e^{-2x}$

$$\Rightarrow \underline{I} = \frac{1}{2}\left(-\frac{1}{2} \sin(2x) e^{-2x} \Big|_0^\infty \right.$$

$$\left. + \int_0^\infty e^{-2x} \cos(2x) \right) dx$$

$$= \frac{1}{2} \int_0^\infty e^{-2x} \cos(2x)\, dx$$

še zadnjič per partes

$u = \cos(2x)$ $\qquad dv = e^{-2x}$

$du = -\sin(2x) \cdot 2$ $\qquad v = -\frac{1}{2} e^{-2x}$

$$\Rightarrow I = \frac{1}{2}\left(-\frac{1}{2} e^{-2x}\cos(2x)\Big|_0^{\infty} - \int_0^{\infty} e^{-2x}\sin(2x)\,dx \right)$$

$$= \frac{1}{4} - \frac{1}{2}\int_0^{\infty} e^{-2x}\sin(2x)\,dx \qquad (1)$$

$$= \frac{1}{2}\int_0^{\infty} e^{-2x}\sin(2x)\,dx$$

$$\Rightarrow \int_0^{\infty} e^{-2x}\sin(2x)\,dx = \frac{1}{4} \qquad (2)$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\infty} e^{-2x}\sin(2x)\,dx = \frac{1}{8}$$

$$\Rightarrow \boxed{\text{volume} = \frac{\pi}{8}} \qquad (2)$$