LINEARNA ALGEBRA 2020/21
26. VAJE: 28. 4. 2021

1. Naj bo A realna antisimetrična matrika. Pokaži, da ima A samo imaginarne lastne vrednosti, matrika $I - A$ je obrnljiva, $U = (I + A)(I - A)^{-1}$ je ortogonalna(unitarna) in A je podobna diagonalni matriki (nad $\mathbb{C}$).

2. Naj bo V vektorski prostor s skalarnim produktom in W podprostor z ortonormirano bazo $\{b_1, \ldots, b_n\}$. Naj bo preslikave $P \colon V \to V$ definirana z

$$Px = \sum_{j=1}^{n} x, b_j b_j$$

   Pokaži, da je $P^2 = P$ in $P^* = P$.

   Naj za preslikavo Q velja $Q^2 = Q$ in $Q^* = Q$. Pokaži, da je $\operatorname{im} Q = \ker Q^{\perp}$.

3. Poišči diagonalizacijo matrike A z ortogonalno prehodno matriko

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{bmatrix}$$

4. Ali je matrika

$$\begin{bmatrix} & & i \\ & 1 & \\ i & & \end{bmatrix}$$

   sebiadjungirana, normalna, unitarna?

5. Določi vse pare vektorjev $a, b \in \mathbb{C}^n$, za katere je preslikava $x \mapsto \langle x, a \rangle b$ normalna, sebiadjungirana.

6. Za normalno matriko A pokaži:

   a) $\ker A = \ker A^*$
   b) $\operatorname{im} A = \operatorname{im} A^*$
   c) Če je $\ker A \subseteq \operatorname{im} A$, potem je A obrnljiva.
   d) Če je $A^2 = 0$, potem je $A = 0$.

7. Naj bo V vektorski prostor nad $\mathbb{C}$. Pokaži, da je $A = 0$ natanko tedaj, ko je $\langle Ax, x \rangle = 0$ za vse $x \in V$. Pokaži, da trditev ne velja za vektorske prostore nad $\mathbb{R}$.

Do sem prejšnjič

1. Naj bo A realna antisimetrična matrika. Pokaži, da ima A samo imaginarne lastne vrednosti, matrika $I-A$ je obrnljiva, $U=(I+A)(I-A)^{-1}$ je ortogonalna(unitarna) in A je podobna diagonalni matriki (nad $\mathbb{C}$). → A je normalna → normalne matrike imajo diagonalizacijo.

$U \in M_n(\mathbb{R})$ ortogonalna $\quad U^T U = UU^T = I$

$U \in M_n(\mathbb{C})$ unitarna $\quad U^H U = UU^H = I$

(stolpci matrike U so ortonormalni)

$$U^H = \left((I+A)(I-A)^{-1}\right)^H = \left((I-A)^{-1}\right)^H (I+A)^H = \star$$

- $(B^{-1})^H = (B^H)^{-1}$ $\qquad \boxed{(B^{-1})^H = B^{-H}}$

$$(B^{-1})^H \cdot B^H = (B \cdot B^{-1})^H = I^H = I$$

$$B^H \cdot (B^{-1})^H = \dots = I$$

$$\star = (I-A)^{H^{-1}} (I+A)^H = (I^H - A^H)^{-1} (I^H + A^H)$$

$$= (I+A)^{-1}(I-A)$$

$A^H = A^T = -A$

$(I-A)(I+A) = (I+A)(I-A)$

zamenjamo

$$U^H U = (I+A)^{-1}(I-A) \cdot (I+A)(I-A)^{-1} = I$$

UU^H – isti postopek

2. Naj bo V vektorski prostor s skalarnim produktom in W podprostor z ortonormirano bazo $\{b_1, \ldots, b_n\}$. Naj bo preslikave $P\colon V \to V$ definirana z

$$Px = \sum_{j=1}^{n} \langle x, b_j \rangle b_j$$

Pokaži, da je $P^2 = P$ in $P^* = P$.

Naj za preslikavo Q velja $Q^2 = Q$ in $Q^* = Q$. Pokaži, da je $\operatorname{im} Q = \ker Q^\perp$.

$$P^2_x = P(P_x) = P\left(\sum_{j=1}^{n} \langle x, b_j \rangle b_j\right) = \sum_{j=1}^{n} \langle x, b_j \rangle P b_j = \circledast$$

$$Pb_j = \sum_{i=1}^{n} \underbrace{\langle b_j, b_i \rangle}_{\delta_{ij}} b_i = b_j$$

$$\circledast = \sum_{j=1}^{n} \langle x, b_j \rangle b_j = P_x \rightsquigarrow P = P^2$$

Od zadnjič $\quad A_x = \langle x, a \rangle b \rightsquigarrow A^*_y = \langle y, b \rangle a$

$+ \quad (A+B)^* = A^* + B^*$

Označimo $\quad P_j x = \langle x, b_j \rangle b_j \quad$ očitno $\quad P_j^* = P_j$

$$P = \sum P_j \longrightarrow P^* = \sum_{j=1}^{n} P_j^* = \sum_{j=1}^{n} P_j = P.$$

Za P velja $\quad V = \operatorname{im} P \oplus \ker P \quad$ in $\quad \operatorname{im} P^\perp = \ker P$

$Q = Q^2$, $Q^* = Q$

(tudi iz nekaj vaj nazaj)

$V = \operatorname{im} Q \oplus \ker Q$

$I = (I - Q) + Q \rightsquigarrow V = \operatorname{im} Q + \operatorname{im} Q$

$\operatorname{im} Q \cap \ker Q = \{0\}$

1. možnost: dimenzijska enačba
2. možnost: direktno $Q^2 x = 0 \Rightarrow Qx = 0$

$Qx \in \ker Q \cap \operatorname{im} Q$

$\operatorname{im} Q^{\perp} = \ker Q^* = \ker Q$

alternativno lahko enakost pokažemo direktno.

3. Poišči diagonalizacijo matrike A z ortogonalno prehodno matriko

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{bmatrix}$$

Vsaka simetrična ($\mathbb{R}$) (hermitska ($\mathbb{C}$)) je ortogonalno (unitarno) podobna diagonalni realni matriki.

Postopek diagonalizacije.

1. Poiščemo bazo v kateri se matrika diagonalizira
2. Dobljeno bazo ortonormiramo z GS postopkom.

$$p_A(\lambda) = \det(A-\lambda I) = \det \begin{bmatrix} 4-\lambda & -1 & 1 \\ -1 & 4-\lambda & -1 \\ 1 & -1 & 4-\lambda \end{bmatrix}$$

$$= \det \begin{bmatrix} 3-\lambda & 0 & \lambda-3 \\ 0 & 3-\lambda & 3-\lambda \\ 1 & -1 & 4-\lambda \end{bmatrix} = (3-\lambda)^2 \det \begin{bmatrix} 1 & & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 4-\lambda \end{bmatrix}$$

$$= -(\lambda-6)(\lambda-3)^2$$

$\ker(A-3I)$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

$$t\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\ker(A-6I)$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\0\\-1\end{bmatrix} \qquad \underline{\begin{bmatrix}1\\-1\\1\end{bmatrix}}$$

$$\boxed{\begin{bmatrix}1\\1\\0\end{bmatrix}} \quad v_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix} - \frac{\left\langle \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\rangle} \begin{bmatrix}1\\1\\0\end{bmatrix} =$$

$$= \begin{bmatrix}1\\0\\-1\end{bmatrix} - \frac{1}{2} \begin{bmatrix}1\\1\\0\end{bmatrix} = \boxed{\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-1\end{bmatrix}}$$

$$\underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix}1\\1\\0\end{bmatrix}}_{v_1}, \quad \sqrt{\frac{1}{4}+\frac{1}{4}+1} = \sqrt{\frac{3}{2}} \qquad \underbrace{\sqrt{\frac{2}{3}} \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\-1\end{bmatrix}}_{v_2} \qquad \underbrace{\frac{1}{\sqrt{3}} \begin{bmatrix}1\\-1\\1\end{bmatrix}}_{v_3}$$

$$U = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$U^{-1} A U = \begin{bmatrix}3 & & \\ & 3 & \\ & & 6\end{bmatrix}$$

$$\boxed{U^{-1} = U^{T}}$$

4. Ali je matrika

$$\begin{bmatrix} & & i \\ & 1 & \\ i & & \end{bmatrix}$$

sebiadjungirana, normalna, unitarna?

$$A^* = A \qquad A^*A = AA^* \qquad A^* = A^{-1}$$

(v našem primeru $A^H = A$)

$$A^H = \begin{bmatrix} 0 & 0 & -i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{bmatrix} \neq A$$

$$\begin{bmatrix} & & i \\ & 1 & \\ i & & \end{bmatrix} \begin{bmatrix} & & -i \\ & 1 & \\ -i & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow A \text{ je unitarna}$$

Unitarne matrike so normalne.

5. Določi vse pare vektorjev $a, b \in \mathbb{C}^n$, za katere je preslikava $x \mapsto \langle x, a\rangle b$ normalna, sebiadjungirana.

$Ax = \langle x, a\rangle b \qquad A^* = \langle x, b\rangle a$

<u>Normalna</u>

$A^*Ax = A^*\langle x,a\rangle b = \langle x,a\rangle A^*b = \langle x,a\rangle\langle b,b\rangle a$

$AA^*x = \langle x,b\rangle Aa = \langle x,b\rangle\langle a,a\rangle b$

$AA^*x = A^*Ax$ za vsak x $\rightsquigarrow$ vstavimo a, b

$$\langle a,a\rangle\langle b,b\rangle a = \langle a,b\rangle\langle a,a\rangle b \qquad (a)$$

$$\langle b,a\rangle\langle b,b\rangle a = \langle b,b\rangle\langle a,a\rangle b \qquad (b)$$

1) $a=0$ ali $b=0$ ✓

2) $a, b \neq 0$

$\langle a,a\rangle \neq 0$ lahko krajšamo

(a) $\rightsquigarrow$ $\langle b,b\rangle a = \langle a,b\rangle b$ $\rightsquigarrow$ a, b sta lin odvisna (ne rabimo obeh enačb)

(b) $\rightsquigarrow$ $\langle b,a\rangle a = \langle a,a\rangle b$

$\langle b,b\rangle$ - krajšamo

$a = \lambda b$ $\overset{\text{Določimo še } \lambda}{\rightsquigarrow}$ $\quad \langle x,\lambda b\rangle\langle b,b\rangle\lambda b = \langle x,b\rangle\langle\lambda b,\lambda b\rangle b$

$\qquad\qquad \lambda\cdot\bar\lambda\langle x,b\rangle\langle b,b\rangle \qquad\qquad \lambda\cdot\bar\lambda\langle x,b\rangle\langle b,b\rangle b$

enakost velja za vse $\lambda \in \mathbb{C}$

<u>Sebi-adjungiranost:</u> so normalne $\rightsquigarrow$ $a = \lambda b$

$\langle x,\lambda b\rangle b = \langle x,b\rangle\lambda b \longrightarrow \bar\lambda = \lambda \rightsquigarrow \lambda \in \mathbb{R}$

$\bar\lambda\langle x,b\rangle b$

6. Za normalno matriko A pokaži:

a) $\ker A = \ker A^*$

b) $\operatorname{im} A = \operatorname{im} A^*$

c) Če je $\ker A \subseteq \operatorname{im} A$, potem je A obrnljiva.

d) Če je $A^2 = 0$, potem je $A = 0$.

a) $x \in \ker A \iff Ax = 0 \iff \|Ax\| = 0$

$$\boxed{\begin{array}{ccc} \|Ax\|^2 & = & \|A^*x\|^2 \\ \| & & \| \\ \langle Ax, Ax\rangle & & \langle A^*x, A^*x\rangle \\ \| & & \| \\ \langle x, A^*Ax\rangle & \underset{\text{normalnost}}{=} & \langle x, AA^*x\rangle \end{array}}$$

$\Updownarrow$

$\|A^*x\| = 0$

$\Updownarrow$

$x \in \ker A^*$

b) $\operatorname{im} A = (\ker A^*)^\perp \underset{a)}{=} \ker A^\perp = \operatorname{im} A^*$

c) kvadratna matrika A je obrnljiva $\iff \ker A = \{0\}$

$\ker A \subseteq \operatorname{im} A = \operatorname{im} A^* = \ker A^\perp \to \ker A \subseteq (\ker A)^\perp \rightsquigarrow \ker A = \{0\}$

$$\boxed{\begin{array}{l} \text{če je } U \subseteq U^\perp \rightsquigarrow U = \{0\} \\ U \subseteq U \cap U^\perp = \{0\} \end{array}}$$

d) $A^2 = 0 \iff \operatorname{im} A \subseteq \ker A$

$$\operatorname{im} A \subseteq \ker A = \operatorname{im} A^{*\perp} = \operatorname{im} A^{\perp} \rightsquigarrow \operatorname{im} A \subseteq \operatorname{im} A^{\perp} \rightsquigarrow \operatorname{im} A = 0 \rightsquigarrow A = 0.$$

Normalne matrike

- matrike, ki so unitarno podobne diagonalnim matrikam

Hermitske matrike

matrike, ki so unitarno podobne realnim diagonalnim matrikam.