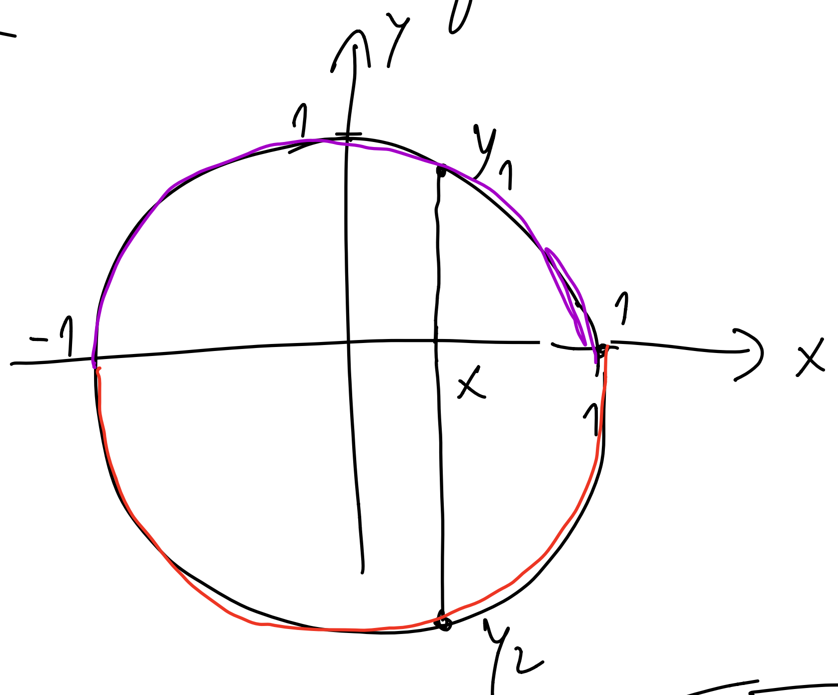
Parametrične krivulje

Param. krivulja:

$$p : I \subseteq \mathbb{R} \to \mathbb{R}^d, \quad d \geq 1$$

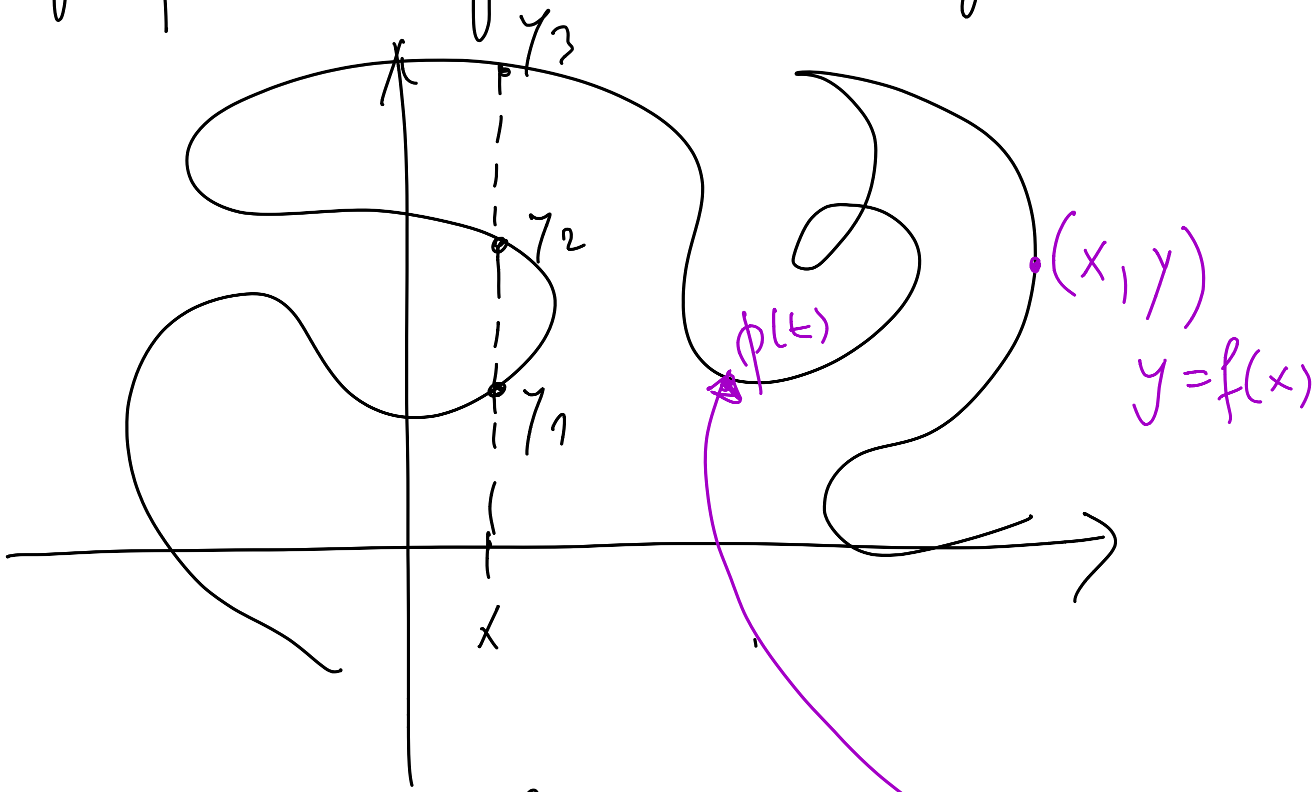
$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_d \end{pmatrix}; \quad p_i : I \to \mathbb{R}, \; i = 1, 2, \ldots, d$$

Uporaba: Risanje krožnice!



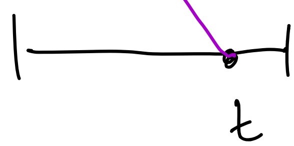
$$x^2 + y^2 = 1 \Rightarrow y = \pm\sqrt{1 - x^2}$$

Kaj pa risanje take krivulje:



$$p : I \longrightarrow \mathbb{R}^2$$

$$t \in I \Rightarrow p(t) \in \mathbb{R}^2 :$$



<u>Parametrizacija</u> krivulje $p(I)$ je <u>regularna</u> preslikava $p : I \to \mathbb{R}^d$.

Regularnost: $p(t) \neq 0, \; \forall t \in I$.

<u>Reparametrizacija</u>:

$$\rho : J \longrightarrow I \;, \; \rho' > 0$$

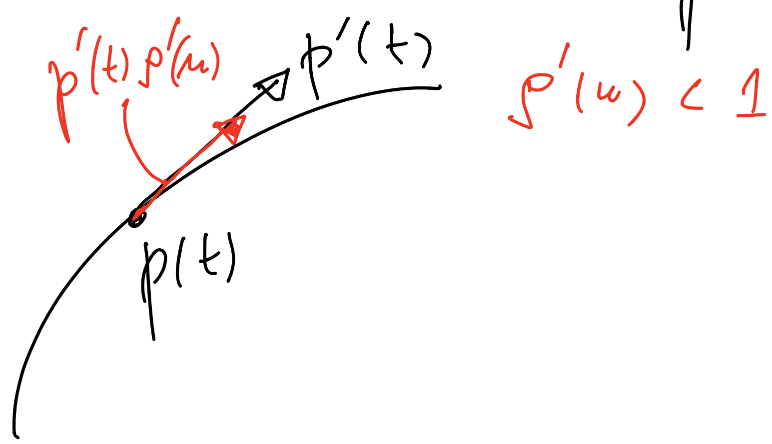
$$\tilde{p} := p \circ \rho : J \longrightarrow \mathbb{R}^d$$

Ali imata p in $\tilde{p}$ isti graf (sliko)? DA, predstavljata isto krivuljo, če je ρ bijekcija.

Torej je slika preslikave $p(I)$ neodvisna od reparametrizacije.

Nekatere količine se pri reparametrizaciji ne ohranjajo: recimo hitrost!

$p'(t)$ ... vektor hitrosti v točki $p(t)$



$\tilde{p} = p \circ \rho$ in $\rho(u) = t$ $\Rightarrow$

$$(p \circ \rho)(u) = p(\rho(u)) = p(t)$$

od reparam. po u

$$(p \circ \rho)'(u) = p'(\rho(u)) \cdot \rho'(u)$$

$$\underbrace{\qquad}_{p'(t)}$$

Katere količine se pa ne spremenijo z reparametrizacijo?!

① Enotski tangentni vektor!

② $\mathbb{R}^2$: fleksijska ukrivljenost:

$$\kappa(t) = \frac{p'(t) \times p''(t)}{\|p'(t)\|^3}$$

planarni vektorski produkt

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$a \times b = a_1 b_2 - a_2 b_1 = \det[a, b]$$

p, $p \circ \rho$ naj bosta reparamet. krivulji

$$\kappa_{p\circ\rho}(u) = \frac{(p\circ\rho)'(u) \times (p\circ\rho)''(u)}{\|(p\circ\rho)'(u)\|^3}$$

$$(p\circ\rho)'(u) = p'(t)\,\rho'(u)$$

$$(p\circ\rho)''(u) = \left(p'(\rho(u)) \cdot \rho'(u)\right)'$$

$$= p''(\rho(u))\rho'(u)^2 + p'(\rho(u))\rho''(u)$$

$$= p''(t)\rho'(u)^2 + p'(t)\rho''(u)$$

$$\kappa_{p\circ\rho}(u) = \frac{p'(t)\rho'(u) \times \left(p''(t)\rho'(u)^2 + p'(t)\rho''(u)\right)}{\|p'(t)\rho'(u)\|^3}$$

$$= \frac{\rho'(u)^3\, p'(t)\times p''(t) + \rho'(u)\rho''(u)\overbrace{p'(t)\times p'(t)}^{0}}{\rho'(u)^3\, \|p'(t)\|^3}$$

$$= \frac{p'(t)\times p''(t)}{\|p'(t)\|^3} = \kappa_p(t)$$

Polinomske parametrične krivulje

$$p: I \subseteq \mathbb{R} \to \mathbb{R}^d; \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_d \end{pmatrix}$$

$p_i : I \to \mathbb{R}$ polinom $\forall i$

Krožnica nima polinomske parametrizacije!

Izkaže se, da je pomembno, kakšna je baza polinomov!

Bézierjeve krivulje

P. E. Bézier (1910 – 1999) inženir pri Renaultu

P. F. de Casteljau (1930 – 1999)

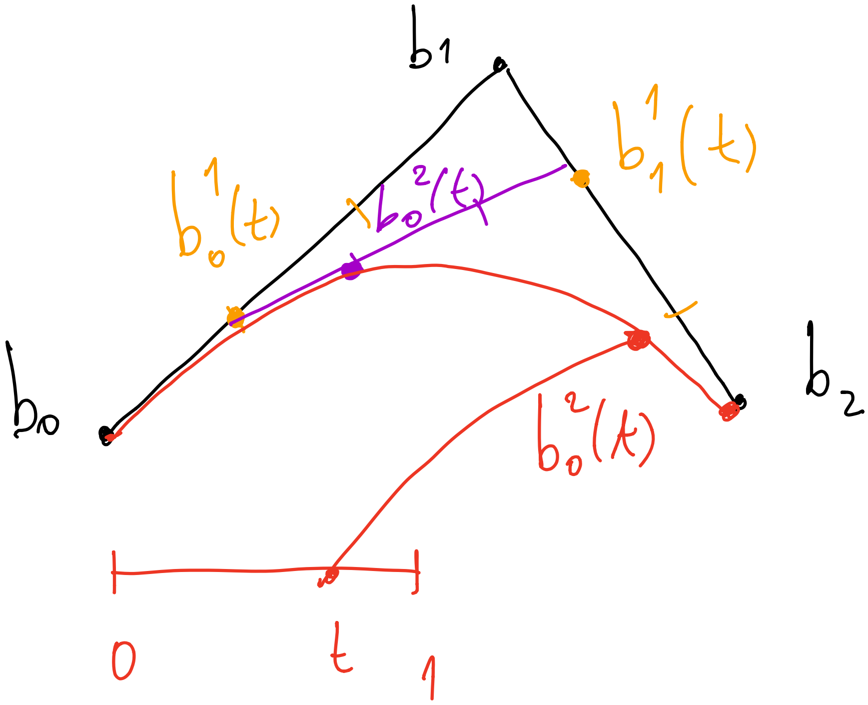
Primer:

Dani so tri točke $b_0, b_1, b_2 \in \mathbb{R}^2$ in $t \in [0,1]$.
Izvedemo naslednji algoritem:

$$b_0^1(t) = (1-t)\, b_0 + t\, b_1,$$

$$b_1^1(t) = (1-t)\, b_1 + t\, b_2,$$

$$\boxed{b_0^2(t)} = (1-t)\, b_0^1(t) + t\, b_1^1(t)$$



$t = \frac{1}{3}$

de Casteljaujev algoritem:

Podatki: $b_j,\ j=0,1,\dots,n$; $t \in [0,1]$

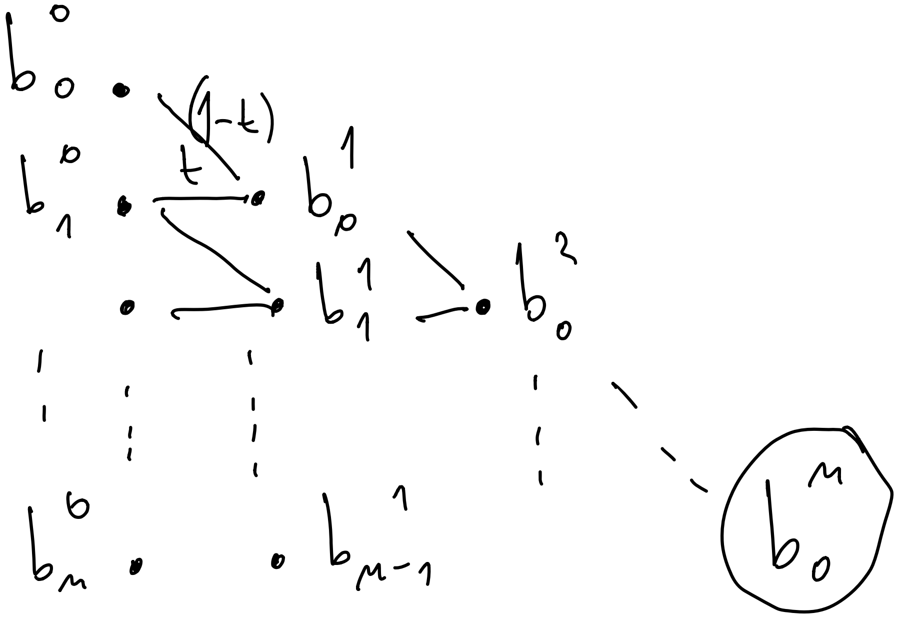
Definiramo: $b_j^0(t) = b_j$, $j = 0,1,\dots,n$

Ponavljamo:

$$b_j^k(t) = (1-t)\, b_j^{k-1}(t) + t\, b_{j+1}^{k-1}(t),$$

$$k = 1,2,\dots,n;\quad j = 0,1,\dots,n-k.$$

Rezultat: $b^n(t) = b_0^n(t)$ točka na t.i. Bézierovi krivulji pri parametru t.



Bernsteinova oblika Béziereve krivulje

Def: Bernsteinov polinom stopnje n z indeksom i je def. kot

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

$$t \in [0,1].$$

Nekatere lastnosti:

- $B_i^n(t) \geq 0$.
- So baza prostora polinomov st. $\leq n$.
- $B_i^n(t) = B_{n-i}^n(1-t)$

- Tvorijo particijo enote:

$$\sum_{i=0}^{n} B_i^n(t) \equiv 1.$$

Res: $\left(t + (1-t)\right)^n$

$$= \sum_{i=0}^{n} \underbrace{\binom{n}{i} t^i (1-t)^{n-i}}_{B_i^n(t)} = 1$$

- Polinom B_i^n ima natanko en ekstrem (max.) v točki i/n.

- Velja rekurzija

$$B_i^n(t) = (1-t) B_i^{n-1}(t) + t B_{i-1}^{n-1}(t)$$

- Odvod:

$$\frac{dB_i^n}{dt}(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$$

Izrek: Naj bodo $b_j \in \mathbb{R}^d$, $j=0,1,\dots,n$, kontrolne točke Bézierjeve krivulje b^n. Potem je

$$b^n(t) = \sum_{j=0}^{n} b_j B_j^n(t).$$