INDUKCIJA

1. S pomočjo indukcije dokaži, da za vsako naravno število n velja

$$2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n+1)}{2}.$$

POPOLNA INDUKCIJA

$T(n)$ ... trditev o naravnem številu $n \in \mathbb{N}$.
S popolno indukcijo dokazujemo, da $T(n)$ velja za $\forall n \in \mathbb{N}$. "za vsak"
Indukcija sestoji iz dveh korakov:

① Baza indukcije: Preverimo, da velja $T(1)$.
($T(n)$ velja za $n = 1$).

② Indukcijski korak: Dokažemo $T(n) \Rightarrow T(n+1)$.
Če velja trditev za nek $n \in \mathbb{N}$, potem velja tudi za naslednje naravno število $n+1$.

Če uspemo izpeljati koraka ① in ②, $T(n)$ velja za $\forall n \in \mathbb{N}$.

Naš primer

$T(n)$: $$\underbrace{2+5+8+\cdots+(3n-1)}_{n=1 \;\to\; 2} = \underbrace{\frac{n(3n+1)}{2}}_{n=1}.$$

① $n = 1$:

$3n-1 = 2 \Longrightarrow 2$

$$\frac{n(3n+1)}{2} \underset{n=1}{=} \frac{1 \cdot 4}{2} = 2 \;\checkmark$$

② I.K.: Želimo izpeljati, da če velja $T(n)$, potem velja tudi $T(n+1)$.

I.P. — indukcijska predpostavka

I.K. — indukcijski korak

Kaj dobimo na levi, če namesto n vstavimo $n+1$?

$$2+5+8+\dots+(3(n+1)-1) = 2+5+8+\dots+(3n-1)+(3(n+1)-1)$$

I.P. $$2+5+8+\dots+3n-1 = \frac{n(3n+1)}{2}$$

$$\boxed{2+5+8+\dots+(3(n+1)-1)} = 2+5+\dots+(3n-1)+(3n+2) =$$

$$= \frac{n(3n+1)}{2} + (3n+2) = \frac{3n^2+n+6n+4}{2} =$$

I.P.

velja

$$= \frac{3n^2+7n+4}{2} \stackrel{?}{=} \boxed{\frac{(n+1)(\overbrace{3(n+1)+1}^{3n+4})}{2}} =$$

n zamenjamo z n+1

$$= \frac{3n^2+7n+4}{2}$$

2. S pomočjo indukcije dokaži, da za vsako naravno število n velja

$$1 \cdot 2^1 + 2 \cdot 2^2 + \cdots n \cdot 2^n = 2^n(2n-2) + 2.$$

I.P.

① $n = 1$: $1 \cdot 2^1 = 2$

$2^1(2 \cdot 1 - 2) + 2 = 2 \cdot 0 + 2 = 2$ ✓

② I.K. $1 \cdot 2^1 + 2 \cdot 2^2 + \ldots + (n+1) \cdot 2^{n+1} = 2^{n+1}(\underbrace{2(n+1) - 2}_{2n+\cancel{2}-\cancel{2} = 2n}) + 2$

ŽELJA

$1 \cdot 2^1 + 2 \cdot 2^2 + \ldots + (n+1) \cdot 2^{n+1} = 1 \cdot 2^1 + \ldots + n \cdot 2^n + (n+1) 2^{n+1} =$

$= 2^n(2n-2) + 2 + (n+1)2^{n+1} =$

I.P.

$= 2^{n+1} n - 2^{n+1} + 2 + (n+1) \cdot 2^{n+1} =$

$= 2^{n+1}(n - \cancel{1} + n + \cancel{1}) + 2 = 2^{n+1}(2n) + 2$ ✓

3. S pomočjo indukcije dokaži, da za vsako naravno število n število 5 deli $n^5 - n$.

$T(n)$: 5 deli $(n^5 - n)$ I.P. (5 deli $\underset{\mathbb{Z}}{n} \iff n = 5\underset{\mathbb{Z}}{k}$)

① $n = 1$: $n^5 - n = 0$

5 deli 0 ✓

$n \to (n+1)$

② I.K.

Želja: 5 deli $((n+1)^5 - (n+1)) =$

$= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + \cancel{1} - n - \cancel{1} \overset{*}{=}$

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

$(a+b)^5 = 1 \cdot \overbrace{a^5}^{a^5} + 5 \cdot a^4 b + 10 a^3 b^2 + 10 a^2 b^3 + 5ab^4 + 1 \cdot \overbrace{b^5}^{b^5}$

$\overset{*}{=} n^5 - n + 5n^4 + 10n^3 + 10n^2 + 5n \overset{\text{I.P.}}{=}$

$$= 5\cdot k + 5n^4 + 10n^3 + 10n^2 + 5n =$$

$$= 5\cdot \underbrace{(k + n^4 + 2n^3 + 2n^2 + n)}_{m} = 5\cdot m$$

4. S pomočjo indukcije dokaži, da za vsako naravno število n je število $1 + 2^{3n+1} + 2^{6n+2}$ deljivo s 7.

7 deli $1 + 2^{3n+1} + 2^{6n+2}$

① $n = 1$: $1 + 2^4 + 2^8 = 1 + 16 + 256 = 273 = 7\cdot 39$

$$\begin{array}{r} 16 \\ 1 \\ \hline 273 \end{array}$$

$273 : 7 = 39$

② $1 + 2^{3(n+1)+1} + 2^{6(n+1)+2}$:

I.P. 7 deli $1 + 2^{3n+1} + 2^{6n+2} \Longrightarrow$

$1 + 2^{3n+1} + 2^{6n+2} = 7\cdot k \Longrightarrow$

$\Longrightarrow 2^{6n+2} = 7k - 2^{3n+1} - 1$

$$1+2^{3(n+1)+1}+2^{6(n+1)+2} = 1+2^{3n+4}+2^{6n+8} =$$

$$= 1+2^{3n+4}+2^{6n+2+6} = 1+2^{3n+4}+2^{6}\cdot 2^{6n+2} =$$

$$= 1+2^{3n+4}+2^{6}\left(7k-2^{3n+1}-1\right) =$$

$\uparrow$

I.P.

$$= 1+2^{3n+4}+7\cdot 2^{6}\cdot k-2^{3n+7}-2^{6} =$$

$$= -63+7\cdot 2^{6}\cdot k+2^{3n+4}(1-2^{3}) =$$ ✓

$$= -7\cdot 9+7\cdot 2^{6}\cdot k+2^{3n+4}(-7) =$$

$$= 7\left(2^{6}\cdot k-9-2^{3n+4}\right) = 7\cdot m$$ ✓

5. S pomočjo indukcije dokaži, da za vsako naravno število n velja

$$\sqrt{n} \leq \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}.$$

I.P.

1) $n = 1$: $\sqrt{n} = \sqrt{1} = 1$

$\frac{1}{\sqrt{1}} + \ldots + \frac{1}{\sqrt{n}} = 1$

$1 \leq 1$ ✓

2) I.K.

$\sqrt{n+1}$: tega izraza ne moremo direktno povezati z I.P., zato si poglejmo desno stran.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \underset{\text{I.P.}}{\geq} \sqrt{n} + \frac{1}{\sqrt{n+1}} \overset{?}{\geq} \underbrace{\sqrt{n+1}}_{\text{ŽELJA}}$$

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} \overset{?}{\geq} \sqrt{n+1} \quad \Big/ \cdot \sqrt{n+1} \;>0$$

$$\sqrt{n}\sqrt{n+1} + \cancel{1} \overset{?}{\geq} n + \cancel{1} \iff \sqrt{n^2+n} \geq n \quad ✓$$

Drug način (brez indukcije):

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}} \geq \overbrace{\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\dots+\frac{1}{\sqrt{n}}}^{n \text{ členov}} = n\cdot\frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

v tej vsoti je $\frac{1}{\sqrt{n}}$ najmanjši člen

Če vse člene nadomestimo z najmanjšim, dobimo manjšo vsoto.

INDUKCIJA (DODATNI PRIMER)

$$\boxed{\forall n \geq 5: \quad \binom{2n}{n} < 4^{n-1}}$$

DOKAŽI

1) $n = 5$: $\binom{k}{l}$ ... binomski koeficient

$$\binom{k}{l} = \frac{k!}{l!\,(k-l)!}$$

$k, l \in \mathbb{N}_0, \; k \geq l$

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!} = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot 2n}{(1\cdot 2\cdot 3\cdot \ldots \cdot n)(1\cdot 2\cdot 3\cdot \ldots \cdot n)}$$

Za $n = 5$:

$$\frac{10!}{5!\,5!} = \frac{\cancel{1\cdot 2\cdot \ldots \cdot 5}\cdot 6\cdot 7\cdot \overset{2}{8}\cdot 9\cdot \overset{2}{10}}{\cancel{(1\cdot 2\cdot \ldots \cdot 5)}\cdot(1\cdot 2\cdot 3\cdot 4\cdot 5)}$$

$$= 7\cdot 2\cdot 9\cdot 2 = 4\cdot 63$$

$$4^{n-1} = 4\cdot 4\cdot 4\cdot 4 = 4\cdot 64$$

$\uparrow$

$n=5$

$$4\cdot 63 < 4\cdot 64 \checkmark$$

2) $\underbrace{\text{I.K.}}$: $\quad$ I.P. $\binom{2n}{n} < 4^{n-1}$

$$\binom{\overbrace{2(n+1)}^{2n+2}}{n+1} = \frac{1\cdot 2\cdot 3\cdot \ldots \cdot 2n\cdot (2n+1)(2n+2)}{(1\cdot 2\cdot \ldots \cdot n\cdot (n+1))(1\cdot 2\cdot \ldots \cdot n\cdot (n+1))}$$

Zgoraj: $\binom{2n}{n} = \frac{1\cdot 2\cdot \ldots \cdot 2n}{(1\cdot 2\cdot \ldots \cdot n)(1\cdot 2\cdot \ldots \cdot n)}$

$$\binom{2(n+1)}{n+1} = \binom{2n}{n}\cdot \frac{(2n+1)(2n+2)}{(n+1)^2} \underset{\text{I.P.}}{<}$$

$$< 4^{n-1}\cdot \frac{(2n+1)(2n+2)}{(n+1)^2} \overset{?}{<} 4^{(n+1)-1} = 4^n$$

ŽELJA

Preverimo: $\cancel{4^{n-1}}\,\frac{(2n+1)(2n+2)}{(n+1)^2} \overset{?}{<} 4^{\cancel{n}} \Big\} : 4^{n-1}$

$$(2n+1)(2n+2) \overset{?}{<} 4\,\overbrace{(n+1)^2}^{n^2+2n+1}$$

$$\cancel{4n^2}+6n+2 \overset{?}{<} \cancel{4n^2}+8n+4$$

$$0 < 2n+2$$

Mimogrede: $(n+1)! = \overbrace{1\cdot 2\cdot 3\cdots n}^{n!}\cdot(n+1)$

$$\boxed{(n+1)! = n!\cdot(n+1)}$$