ZAPOREDJA

2. Podano je zaporedje $\{a_n\}_{n\in\mathbb{N}}$ s splošnim členom

$$a_n = \frac{n^2}{1+n}.$$

(a) Obravnavaj naraščanje in padanje zaporedja $\{a_n\}_{n\in\mathbb{N}}$.

(b) Obravnavaj omejenost zaporedja $\{a_n\}_{n\in\mathbb{N}}$.

(c) Ali zaporedje $\{a_n\}_{n\in\mathbb{N}}$ konvergira?

(a) $a_1 = \frac{1}{2}, \; a_2 = \frac{4}{3}, \; a_3 = \frac{9}{4}, \ldots$

$\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$

Opazimo, da se členi večajo ($a_n < a_{n+1}$), tj. zaporedje <u>(strogo) raste</u>:

<u>$a_n < a_{n+1}$</u>:

$$\frac{n^2}{1+n} \overset{?}{<} \frac{(n+1)^2}{1+(n+1)}$$

$$\frac{n^2}{1+n} \overset{?}{<} \frac{n^2+2n+1}{2+n} \quad \Big/ \cdot (n+1)(n+2)$$

$$\boxed{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}$$

$$n^2(n+2) \overset{?}{<} (n+1)^3$$

$$\cancel{n^3} + 2n^2 \overset{?}{<} \cancel{n^3} + 3n^2 + 3n + 1$$

$$0 \overset{?}{<} n^2 + 3n + 1 \checkmark$$

(b) Zaporedje $\{a_n\}_{n\in\mathbb{N}}$ je navzdol/navzgor omejeno, če obstaja m, da je

$$a_n \geq m \ / \ a_n \leq m$$

- Naše zaporedje je navzdol omejeno z $\frac{1}{2}$ (prvi člen), ker je naraščajoče.
- Naše zaporedje ni navzgor omejeno, ker je stopnja števca ("n^2") večja od stopnje imenovalca ("n"):

$$a_n = \frac{n^2}{n+1} \leq M$$ (recimo, da je navzgor omejeno)

$$\frac{n^2}{n+1} \leq M \Big/ \cdot (n+1)$$

$$n^2 \leq M\cdot(n+1) \iff n^2 - Mn - M \leq 0 \quad \forall n$$

Ker za velike n prevlada vodilni člen (n^2), ki je navzgor neomejen, neenakost ne drži za vsak n. ⟵

(c) Zaporedje a_n konvergira, če obstaja njegova limita, tj.

$$a = \lim_{n\to\infty} a_n \in \mathbb{R}$$

$\boxed{\text{Trditev. Če je zaporedje konvergentno, potem je omejeno (navzgor in navzdol).}}$

$\Downarrow$

Naše zaporedje ni konvergentno, ker ni navzgor omejeno.

3. Podano je zaporedje $\{a_n\}_{n\in\mathbb{N}}$ s splošnim členom

$$a_n = \frac{1-n^2}{1+n^2}.$$

(a) Obravnavaj naraščanje in padanje zaporedja $\{a_n\}_{n\in\mathbb{N}}$.

(b) Obravnavaj omejenost zaporedja $\{a_n\}_{n\in\mathbb{N}}$.

(c) Ali zaporedje $\{a_n\}_{n\in\mathbb{N}}$ konvergira? Izračunaj $\lim_{n\to\infty} a_n$.

(a) $0, -\frac{3}{5}, -\frac{8}{10}, -\frac{15}{17}, \dots$

Zaporedje pada: $a_n > a_{n+1}$

$$\frac{1-n^2}{1+n^2} \overset{?}{>} \frac{1-(n+1)^2}{1+(n+1)^2} \quad \Big/ \cdot (1+n^2)(1+(n+1)^2)$$

$$(1-n^2)(1+(n+1)^2) \overset{?}{>} (1+n^2)(1-(n+1)^2)$$

$$\cancel{1}+(n+1)^2 - n^2 - \cancel{n^2(n+1)^2} \overset{?}{>} \cancel{1} - (n+1)^2 + n^2 - \cancel{n^2(n+1)^2}$$

$$\cancel{n^2}+2n+1-\cancel{n^2} \overset{?}{>} -\cancel{n^2}-2n-1+\cancel{n^2}$$

$$2n+1 \overset{?}{>} -2n-1 \ \checkmark$$

2. način

$$\frac{1-n^2}{1+n^2} = \frac{1}{1+n^2} - \frac{n^2}{1+n^2} = \frac{1}{1+n^2} - \frac{(n^2+1)-1}{1+n^2} =$$

$\nearrow 1$

$$= \frac{1}{1+n^2} - 1 + \frac{1}{1+n^2} = -1 + \frac{2}{n^2+1}$$

PADA

RASTE

PADA $\checkmark$

(b) • Zaporedje pada zato je $a_1 = 0 = \max a_n$ zgornja meja $\Rightarrow$ a_n je navzgor omejeno.

• Spodnja meja:

Domnevamo, da je $-1 < a_n$ za $\forall n$.

$$-1 \overset{?}{<} \frac{1-n^2}{1+n^2} \; \Big\} \cdot (1+n^2) \iff -1-\cancel{n^2} \overset{?}{<} 1-\cancel{n^2} \; ✓$$

(c) Ali a_n konvergira?

$$a_n = \frac{1-n^2 \;\; :n^2}{1+n^2 \;\; :n^2} = \frac{\overbrace{\frac{1}{n^2}}^{\to 0} - 1}{\underbrace{\frac{1}{n^2}}_{\substack{n\to\infty \\ \to 0}} + 1} \xrightarrow{n\to\infty} -1$$

Po definiciji se prepričajmo, da je

$$\lim_{n\to\infty} a_n = -1$$

DEF. $\lim_{n\to\infty} a_n = a$, če za $\forall \varepsilon > 0 \; \exists n_0$:

$$n \geq n_0 \Longrightarrow \underbrace{|a_n - a| < \varepsilon}_{a_n \in (a-\varepsilon, a+\varepsilon)}$$

Zapišimo odstopanje $|a_n - a|$:

$$|a_n - (-1)| = \left|\frac{1-n^2}{1+n^2} + 1\right| = \left|\frac{1-\cancel{n^2}+\cancel{n^2}+1}{n^2+1}\right| =$$

$$= \left|\underbrace{\frac{2}{n^2+1}}_{>0}\right| = \frac{2}{n^2+1} < \varepsilon$$

$$\frac{2}{n^2+1} < \varepsilon \; \Big\} (n^2+1) \iff 2 < \varepsilon(n^2+1) \iff$$

$$\iff 2 - \varepsilon < \varepsilon n^2 \; \Big\} : \underset{\substack{\vee \\ 0}}{\varepsilon} \iff \underbrace{\frac{2-\varepsilon}{\varepsilon} < n^2}_{(*)}$$

Neenačba (*) zagotovo velja za velike n, torej za $n \geq n_0$. $\Longrightarrow \lim_{n\to\infty} a_n = -1$ ✓

Limita na drug način:

Naše zaporedje je padajoče in navzdol omejeno z -1.

Trditev. Če je a_n padajoče / naraščajoče in navzdol / navzgor omejeno, potem je konvergentno in velja

$$\lim_{n\to\infty} a_n = \inf a_n \,/\, \sup a_n$$

D.N. Dokaži, da je $\inf a_n = -1$. ✓

Od tod pa po trditvi velja $\lim_{n\to\infty} a_n = -1$

4. Po definiciji preveri, da velja:

   (a) $\lim_{n\to\infty} \frac{n+2}{2n+1} = \frac{1}{2}$,

   (b) $\lim_{n\to\infty} \frac{n^2+1}{n^2+n} = 1$.

   V obeh primerih določi kateri členi zaporedja se razlikujejo od limite za največ $\frac{1}{10}$.

(a) $\lim_{n\to\infty} \frac{n+2}{2n+1} = \frac{1}{2}$

$$\lim_{n\to\infty} a_n = a \overset{DEF}{\Longleftrightarrow} \forall \varepsilon > 0 \, \exists n_0 : n \geq n_0 \Longrightarrow |a_n - a| < \varepsilon$$

Naš primer: $|a_n - a| = \left|\frac{n+2}{2n+1} - \frac{1}{2}\right| = \left|\frac{\cancel{2n}+4-\cancel{2n}-1}{2(2n+1)}\right| =$

$$= \left|\frac{3}{2(2n+1)}\right| = \frac{3}{2(2n+1)} < \varepsilon \Longleftrightarrow$$

$\frac{3}{2(2n+1)} > 0$

$$\Longleftrightarrow 3 < 2\varepsilon(2n+1) \Longleftrightarrow \frac{3}{2\varepsilon} < 2n+1 \Longleftrightarrow$$

$2\varepsilon > 0$

$$\Longleftrightarrow \frac{3}{2\varepsilon} - 1 < 2n \Longleftrightarrow \frac{3}{4\varepsilon} - \frac{1}{2} < n$$ ✓

Ker je $\mathbb{N}$ navzgor neomejena $\exists n_0 \in \mathbb{N}$:

$n_0 > \frac{3}{4\varepsilon} - \frac{1}{2}$ za $n \geq n_0 \Rightarrow n > \frac{3}{4\varepsilon} - \frac{1}{2} \Rightarrow$

$\Rightarrow |a_n - a| < \varepsilon$. ✓

Za $\varepsilon = \frac{1}{10}$ (drugi del naloge) dobimo

$n_0 > \frac{3}{4 \cdot \frac{1}{10}} - \frac{1}{2} = \frac{30}{4} - \frac{1}{2} = \frac{14}{2} = 7$

$\Rightarrow n_0 = 8$

Razlika $|a_n - a| < \frac{1}{10}$ od 8. člena dalje.

(b) D. N.

6. Izračunaj naslednje limite.

(a) $\lim_{n\to\infty} \frac{n+1}{3n-2}$,

(b) $\lim_{n\to\infty} \frac{2n^2+3n-5}{1-4n^2}$,

(c) $\lim_{n\to\infty} \frac{\sqrt{n}}{n+2}$,

(d) $\lim_{n\to\infty} \frac{2n^3-2n^2+n+6}{n^2(n+1)}$,

(e) $\lim_{n\to\infty} \frac{\sqrt{2n^2+3n+1}-\sqrt{n^2+1}}{n}$,

(f) $\lim_{n\to\infty} \frac{n^2-n}{\sqrt{4n^4+n^2-1}}$,

(g) ... (i)

(a) $\lim_{n\to\infty} \frac{n+1 \;:n}{3n-2 \;:n} = \lim_{n\to\infty} \frac{1+\overbrace{\frac{1}{n}}^{\to 0}}{3-\underbrace{\frac{2}{n}}_{\to 0}} = \frac{1}{3}$

(b) $\lim_{n\to\infty} \frac{2n^2+3n-5 \;: n^2}{1-4n^2 \;: n^2} = \lim_{n\to\infty} \frac{2+\overbrace{\frac{3}{n}}^{\to 0}-\overbrace{\frac{5}{n^2}}^{\to 0}}{\underbrace{\frac{1}{n^2}}_{\to 0}-4} = -\frac{1}{2}$

$$\boxed{\text{Za } \alpha > 0: \lim_{n\to\infty} \frac{c}{n^\alpha} = 0}$$

(c) $\lim_{n\to\infty} \frac{\overbrace{\sqrt{n}}^{=n^{1/2}} \;:n}{n+2 \;:n} = \lim_{n\to\infty} \frac{\overbrace{n^{-1/2}}^{=\frac{1}{n^{1/2}} \to 0}}{1+\underbrace{\frac{2}{n}}_{\to 0}} = \frac{0}{1} = 0$

(d) $\lim_{n\to\infty} \frac{2n^3-2n^2+n+6 \;: n^3}{\underbrace{n^2(n+1)}_{n^3+n^2} \;: n^3} = \lim_{n\to\infty} \frac{2-\overbrace{\frac{2}{n}}^{\to 0}+\overbrace{\frac{1}{n^2}}^{\to 0}+\overbrace{\frac{6}{n^3}}^{\to 0}}{1+\underbrace{\frac{1}{n}}_{\to 0}} = 2$

(e) $\lim_{n\to\infty} \frac{\sqrt{2n^2+3n+1}-\sqrt{n^2+1}}{n} \cdot \frac{(\sqrt{2n^2+3n+1}+\sqrt{n^2+1})}{(\sqrt{2n^2+3n+1}+\sqrt{n^2+1})} =$

$$= \lim_{n\to\infty} \frac{\cancel{2}n^2+3n+\cancel{1}-(\cancel{n^2}+\cancel{1})}{n\,(\sqrt{2n^2+\dots}+\sqrt{n^2+1})} = \lim_{n\to\infty} \frac{\cancel{n}(n+3)}{\cancel{n}(\sqrt{--}+\sqrt{--})} =$$

$(a-b)(a+b)=a^2-b^2$

$$= \lim_{n\to\infty} \frac{n+3 \quad :n}{\sqrt{2n^2+3n+1}+\sqrt{n^2+1} \quad :n} = \lim_{n\to\infty} \frac{1+\overbrace{\frac{3}{n}}^{\to 0}}{\sqrt{2+\underbrace{\frac{3}{n}}_{\to 0}+\underbrace{\frac{1}{n^2}}_{\to 0}}+\sqrt{1+\underbrace{\frac{1}{n^2}}_{\to 0}}} = \frac{1}{\sqrt{2}+1}$$

$$\sqrt{2n^2+3n+1} : n = \frac{\sqrt{2n^2+3n+1}}{n} = \sqrt{\frac{2n^2+3n+1}{n^2}}$$

Če $\sqrt{---}$ delimo z n, moramo pod korenom deliti z n^2.

Krajši način: $\frac{\sqrt{2n^2+3n+1}}{n} - \frac{\sqrt{n^2+1}}{n} =$

$$= \sqrt{2+\underbrace{\frac{3}{n}}_{0}+\underbrace{\frac{1}{n^2}}_{0}} - \sqrt{1+\underbrace{\frac{1}{n^2}}_{0}} \longrightarrow \sqrt{2}-1$$

(f) $\lim_{n\to\infty} \frac{n^2-n \quad :n^2}{\sqrt{4n^4+n^2-1} \quad :n^2} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{\sqrt{(\dots\dots):n^4}} =$

$$= \lim_{n\to\infty} \frac{1-\overbrace{\frac{1}{n}}^{\to 0}}{\sqrt{4+\underbrace{\frac{1}{n^2}-\frac{1}{n^4}}_{0}}} = \frac{1}{2}$$

(g) $\lim_{n\to\infty} \frac{\overbrace{\sqrt{n^5+2n+n^2}}^{\approx n^{\frac{5}{2}}} \quad : n^{5/2}}{\underbrace{\underbrace{\sqrt{n+1}}_{\approx n^{\frac{1}{2}}}\underbrace{(n^2+3n-1)}_{\approx n^2}}_{\text{skupna stopnja } n^{\frac{5}{2}}} \quad : n^{5/2}}$ $\qquad \sqrt{n^5} = n^{\frac{5}{2}}$

$$\lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n^4}}+\frac{1}{\sqrt{n}}}{\left(\sqrt{n+1} : n^{1/2}\right)\cdot\left((n^2+3n-1) : n^2\right)} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{2}{n^4}}+\frac{1}{\sqrt{n}} \to 0}{\sqrt{1+\frac{1}{n}}\cdot\left(1+\frac{3}{n}-\frac{1}{n^2}\right)} = 1$$

$\to 0$

(h) $\lim_{n\to\infty} \frac{2^n+6^n-7^{n+1}}{48\cdot 7^{n-1}-5^n}$

V imenovalcu poiščemo člen z največjo močjo. Eksponentna fja z večjo osnovo ima večjo moč, zato izberemo 7^{n-1}.

Pozor 7^n, $\boxed{5^{n^2}=(5^n)^n}$

MOČNEJŠI

Delimo s 7^n na obeh straneh:

$$\frac{2^n+6^n-7^{n+1} \;:7^n}{48\cdot 7^{n-1}-5^n \;:7^n} = \frac{\left(\frac{2}{7}\right)^n+\left(\frac{6}{7}\right)^n-7}{48/7-\left(\frac{5}{7}\right)^n} \to \frac{-7}{\frac{48}{7}} = -\frac{49}{48}$$

$\to 0$ $\to 0$ $\to 0$

$$\boxed{\text{Za } a\in(-1,1):\ \lim_{n\to\infty} a^n = 0}$$

(i) $\lim_{n\to\infty} \frac{2^n-2^{n+1} \;:2^{n+3}}{2^{n+3}-2^{n-1} \;:2^{n+3}} = \lim_{n\to\infty} \frac{\frac{1}{8}-\frac{1}{4}\cdot 16}{1-\frac{1}{16}\cdot 16} =$

$$= \frac{2-4}{16-1} = -\frac{2}{15}$$

(g) $\lim_{n\to\infty} \frac{\sqrt{n^5+2n}+n^2}{\sqrt{n+1}(n^2+3n-1)}$,

(h) $\lim_{n\to\infty} \frac{2^n+6^n-7^{n+1}}{48\cdot 7^{n-1}-5^n}$,

(i) $\lim_{n\to\infty} \frac{2^n-2^{n+1}}{2^{n+3}-2^{n-1}}$.

7. Izračunaj naslednje limite.

(a) $\lim_{n\to\infty} \left(n^2(\sqrt{n^2-1}-n^3)\right)$,

(b) $\lim_{n\to\infty} \frac{\sqrt{n^2+7n+1}-n}{\sqrt{n^2+2n}-n}$,

(c) $\lim_{n\to\infty} \frac{\sqrt{n+2}-\sqrt{n-5}}{\sqrt{n-3}-\sqrt{n+1}}$.

$n^2 : n^2(\cdots) : n$

(a) $$\frac{n^2(\sqrt{n^2-1}-n^3)(\sqrt{n^2-1}+n^3)}{\sqrt{n^2-1}+n^3} = \frac{n^2(n^2-1-n^6) \;:n^3}{n^3+\sqrt{n^2-1} \;:n^3} =$$

$$= \frac{n-\frac{1}{n}-n^5}{1+\sqrt{\frac{1}{n^4}-\frac{1}{n^6}}} \xrightarrow[n\to\infty]{} \frac{-\infty}{1} = -\infty$$

$\to 0$ $\to -\infty$ $\to 0$

Diskusija

$$\frac{n-\frac{1}{n}-n^5}{1+\sqrt{\frac{1}{n^4}-\frac{1}{n^6}}} = \underbrace{\frac{n}{1+\sqrt{\ldots}}}_{\downarrow\ \infty} - \underbrace{\frac{\frac{1}{n}}{1+\sqrt{\ldots}}}_{\downarrow\ 0} - \underbrace{\frac{n^5}{1+\sqrt{\ldots}}}_{\downarrow\ -\infty}$$

$$\boxed{\infty+\infty=\infty}$$

$$\boxed{\infty-\infty \quad \text{nedefinirano}} : \quad \text{"}n-n=0\text{"}$$

$$\text{"}n^2-n=\infty\text{"}$$

(b) $\lim_{n\to\infty} \frac{\sqrt{n^2+7n+1}-n}{\sqrt{n^2+2n}-n}$

$$\frac{(\sqrt{n^2+7n+1}-n)(\sqrt{n^2+7n+1}+n)(\sqrt{n^2+2n}+n)}{(\sqrt{n^2+2n}-n)(\sqrt{n^2+7n+1}+n)(\sqrt{n^2+2n}+n)} =$$

$$= \frac{(\cancel{n^2}+7n+1-\cancel{n^2})(\sqrt{n^2+2n}+n) : n^2}{(\cancel{n^2}+2n-\cancel{n^2})(\sqrt{n^2+7n+1}+n) : n^2} =$$

$$= \frac{(7+\overset{\to 0}{\frac{1}{n}})(\sqrt{1+\overset{\to 0}{\frac{2}{n}}}+1)}{2(\sqrt{1+\underset{\searrow 0}{\frac{7}{n}}+\underset{\swarrow}{\frac{1}{n^2}}}+1)} \xrightarrow{n\to\infty} \frac{7}{2}$$

$$(\quad)\cdot(\quad) : n^2 = ((\quad):n)((\quad):n)$$

(c) $\lim_{n\to\infty} \frac{\sqrt{n+2}-\sqrt{n-5}}{\sqrt{n-3}-\sqrt{n+1}}$.

$$\frac{(\sqrt{n+2}-\sqrt{n-5})(\sqrt{n+2}+\sqrt{n-5})(\sqrt{n-3}+\sqrt{n+1})}{(\sqrt{n-3}-\sqrt{n+1})(\sqrt{n+2}+\sqrt{n-5})(\sqrt{n-3}+\sqrt{n+1})} =$$

$$= \frac{(\cancel{n}+2-(\cancel{n}-5))(\sqrt{n-3}+\sqrt{n+1})}{(\cancel{n}-3-(\cancel{n}+1))(\sqrt{n+2}+\sqrt{n-5})} =$$

$$= \frac{7(\sqrt{n-3}+\sqrt{n+1}) : \sqrt{n}}{-4(\sqrt{n+2}+\sqrt{n-5}) : \sqrt{n}} = -\frac{7}{4}\,\frac{\sqrt{1-\frac{3}{n}}+\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{2}{n}}+\sqrt{1-\frac{5}{n}}} \xrightarrow{n\to\infty} -\frac{7}{4}$$

0

1. Poišči vsa stekališča zaporedja $\{a_n\}_{n\in\mathbb{N}}$, podanega s splošnim členom

   (a) $a_n = \frac{(-1)^n \sin(n)}{n+1}$,

   (b) $a_n = 1 + (-1)^{n+1} \sin\left(\frac{n\pi}{4}\right)$.

(a) $a_n = \frac{(-1)^n \sin n}{n+1}$

$s \in \mathbb{R}$ je stekališče zaporedja $\{a_n\}_n$, če za $\forall \varepsilon > 0$ v okolici $(s-\varepsilon, s+\varepsilon)$ najdemo neskončno členov zaporedja.

Trditev: Če je zaporedje konvergentno in je $\lim_{n\to\infty} a_n = a$, potem je a edino stekališče.

Trditev: Če je $s \in \mathbb{R}$ stekališče, je s limita nekega podzaporedja: $n_1 < n_2 < n_3 < \ldots$ in $s = \lim_{k\to\infty} a_{n_k}$.

Naš primer: $a_n = \frac{(-1)^n \sin n}{n+1}$

$\sin n \xrightarrow{n\to\infty}$ ima vrednosti med -1 in 1

OMEJENO

$n+1 \xrightarrow{n\to\infty} \infty$

$\lim_{n\to\infty} a_n = 0 \implies 0$ je edino stekališče

Trditev: Če je $\lim_{n\to\infty} b_n = 0$ in je c_n omejeno zaporedje, potem je

$$\boxed{\lim_{n\to\infty} \underbrace{b_n}_{\to 0} \cdot \underbrace{c_n}_{\text{omejeno}} = 0}$$

V našem primeru: $\underbrace{\frac{1}{n+1}}_{\to 0} \cdot \underbrace{(-1)^n \sin n}_{\text{omejeno na } [-1,1]} \longrightarrow 0$

(b) $a_n = 1 + (-1)^{n+1} \sin\left(\frac{n\pi}{4}\right)$.

Opazujmo $\sin\left(\frac{n\pi}{4}\right)$: $\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0$, se ponavlja

$n=1$, $n=2$, $n=3$, $n=4$, $n=5$, $n=6$, $n=7$, $n=8$