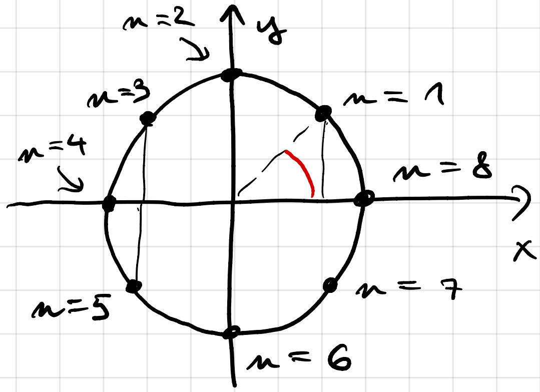

① Za $n = 4k$ je $\sin\frac{n\pi}{4} = 0 \Rightarrow$

$\Rightarrow a_n = 1$

$a_{\underset{n_k}{4k}} = 1$! $\boxed{\lim\limits_{k\to\infty} a_{4k} = \boxed{1}}$

② Za $n = 2 + 8k$ (sodo) je $\sin\frac{n\pi}{4} = 1 \Rightarrow$

$\Rightarrow a_n = 1 + (-1)^{n+1} \cdot 1 = 1 - 1 = 0$

$n+1$: liho

$$\boxed{\lim_{k\to\infty} a_{2+8k} = \boxed{0}}$$

③ Za $n = 6 + 8k$ (sodo) je $\sin\frac{n\pi}{4} = -1 \Rightarrow$

$\Rightarrow a_n = 1 + \underbrace{(-1)^{n+1}}_{-1} \cdot (-1) = 1 + 1 = 2$

$$\boxed{\lim_{k\to\infty} a_{6+8k} = \boxed{2}}$$

④ Za $n = 1 + 8k$ (liho) ali $n = 3 + 8k$ (liho) je $\sin\frac{n\pi}{4} = \frac{\sqrt{2}}{2} \Rightarrow$

$\Rightarrow a_n = 1 + (-1)^{n+1}\frac{\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2}$

$n+1$: sodo

$$\boxed{\lim_{k\to\infty} a_{1+8k} = \lim_{k\to\infty} a_{3+8k} = \boxed{1 + \frac{\sqrt{2}}{2}}}$$

⑤ Za $n = 5 + 8k$ (liho) ali $n = 7 + 8k$ (liho) je $\sin\frac{n\pi}{4} = -\frac{\sqrt{2}}{2}$

$\Rightarrow a_n = 1 + \underbrace{(-1)^{n+1}}_{= 1}\left(-\frac{\sqrt{2}}{2}\right) = 1 - \frac{\sqrt{2}}{2}$

$$\boxed{\lim_{k\to\infty} a_{5+8k} = \lim_{k\to\infty} a_{7+8k} = \boxed{1 - \frac{\sqrt{2}}{2}}}$$

V okvirčkih $\boxed{}$ so vsa stekališča $\{a_n\}_n$.

Trditev Če zaporedje razbijemo na končno podzaporedij z limitami $s_1, \dots, s_m$ in pri tem zajamemo vse člene zaporedja, so $s_1, \dots, s_m$ natanko vsa stekališča.

8. Dano je zaporedje s prvim členom $a_1 = 0$ in rekurzivno formulo

$$a_{n+1} = \frac{3a_n + 2}{5}. \quad (R)$$

Pokaži, da je zaporedje monotono in omejeno in izračunaj njegovo limito.

$$\underset{a_1}{0}, \underset{a_2}{\frac{2}{5}}, \frac{16}{25}, \frac{98}{5^3}, \dots$$

$$a_2 \overset{(R)}{=} \frac{3a_1+2}{5} \overset{a_1=0}{=} \frac{3\cdot 0+2}{5} = \frac{2}{5}, \quad a_3 = \frac{3\cdot\frac{2}{5}+2}{5} = \frac{16}{25}$$

$$a_4 = \frac{3\cdot\frac{16}{25}+2}{5} = \frac{98}{5^3}$$

Iz prvih členov sklepamo, da a_n narašča:

- $a_{n+1} > a_n$:

$$\frac{3a_n+2}{5} \overset{?}{>} a_n \quad \Big\} \cdot 5$$

$$3a_n + 2 \overset{?}{>} 5a_n \quad \Big\} -3a_n$$

$$2 \overset{?}{>} 2a_n \quad \Big\} : 2$$

$$1 \overset{?}{>} a_n$$

Torej najprej preverimo omejenost:

- $a_n < 1$:

Indukcija

① $n=1$: $a_1 = 0 < 1$ ✓

② I.K.: $a_{n+1} < 1$

$$(R) \Longrightarrow a_{n+1} = \frac{3a_n+2}{5} < 1 \quad \Big\} \cdot 5$$

$$3a_n + 2 \overset{?}{<} 5 \Big\} -2 \iff 3a_n \overset{?}{<} 3 \overset{:3}{\iff} a_n < 1$$ ✓

INDUKCIJSKA HIPOTEZA

Ker smo ugotovili, da je $a_n < 1$ za $\forall n$ velja tudi naraščanje.

Zaporedje a_n narašča in je navzgor omejeno $\overset{\text{TEORIJA}}{\Longrightarrow}$ a_n konvergira.

TRDITEV Če zaporedje raste/pada in je navzgor/navzdol omejeno, potem konvergira.

Naš primer

Vemo, da obstaja $\lim_{n\to\infty} a_n = a$.

$$a_{n+1} = \frac{3a_n + 2}{5}, \quad a_n \to a$$

$$\downarrow_{n\to\infty} \qquad \downarrow$$

$$a = \frac{3a+2}{5}$$

$$5a = 3a + 2 \Longrightarrow 2a = 2 \Longrightarrow a = 1$$

VAJE 20

9. Dano je zaporedje z rekurzivno formulo

$$a_{n+1} = \frac{a_n}{a_n + 1}. \quad (R)$$

Pokaži, da je zaporedje konvergentno za poljubno izbiro prvega člena in izračunaj njegovo limito. Naj bo $a_1 = a$ prvi člen zaporedja. Določi splošni člen zaporedja $\{a_n\}_{n\in\mathbb{N}}$.

$$\boxed{a_1 = a} \qquad \boxed{a_2 \overset{(R)}{=} \frac{a_1}{a_1+1} = \frac{a}{a+1}} \qquad \boxed{a_3 \overset{(R)}{=} \frac{a_2}{a_2+1}} = \frac{\frac{a}{a+1}}{\left(\frac{a}{a+1}+1\right)} \cdot \frac{(a+1)}{(a+1)} =$$

$$= \frac{a}{a+a+1} = \boxed{\frac{a}{2a+1}}$$

$$a_{\boxed{4}} = \frac{\frac{a}{2a+1}}{\frac{a}{2a+1}+1} = \frac{a}{a+2a+1} = \boxed{\frac{a}{\boxed{3}a+1}}$$

Domneva $a_n = \dfrac{a}{(n-1)a+1}$

INDUKCIJA

① $n=1$: $a_1 = a$ $\quad \dfrac{a}{(n-1)a+1} \overset{n=1}{=} \dfrac{a}{0\cdot a+1} = a$ ✓

② I.K.:

$$a_{n+1} = \frac{a}{na+1}$$

$$a_{n+1} \underset{(R)}{=} \frac{a_n}{a_n+1} \underset{\text{I.P.}}{=} \frac{\frac{a}{(n-1)a+1}}{\frac{a}{(n-1)a+1}+1} =$$

I.P. $a_n = \dfrac{a}{(n-1)a+1}$

$$= \frac{a}{a+(n-1)a+1} = \frac{a}{na+1}$$ ✓

Ugotovili smo, da je

$$a_n = \frac{a}{(n-1)a+1}$$

Če želimo zagotoviti neničelnost imenovalca, dobimo zahtevo: $(n-1)a+1 \neq 0 \iff$

$\iff a \neq -\dfrac{1}{n-1}$ za $\forall n \in \mathbb{N} \iff a \notin \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a}{\underbrace{\underbrace{(n-1)}_{\to\infty}a+1}_{\to\pm\infty\ (\text{če } a\neq 0)}} \underset{\text{za } a\neq 0}{=} \text{"}\frac{a}{\pm\infty}\text{"} = 0$$

Za $a=0$: $a_n = 0 \implies \lim_{n\to\infty} a_n = 0$ ✓

10. Dano je zaporedje s prvima členoma $a_1 = 0$ in $a_2 = 1$ in rekurzivno formulo

$$a_{n+1} = \sqrt{a_n} + \sqrt{a_{n-1}}, \qquad (R)$$

za $n \geq 2$. Pokaži, da je zaporedje monotono in omejeno in izračunaj njegovo limito.

(R) je dvočlena rekurzivna formula, ker je a_{n+1} izraža s prejšnjima dvema členoma.

$$a_3 \overset{(R)}{=} \sqrt{a_2} + \sqrt{a_1} = \sqrt{1} + \sqrt{0} = 1$$

$$a_4 = \sqrt{a_3} + \sqrt{a_2} = \sqrt{1} + \sqrt{1} = 2$$

$$a_5 = \sqrt{a_4} + \sqrt{a_3} = \sqrt{2} + \sqrt{1} = 1 + \sqrt{2}$$

$\vdots$

$0, 1, 1, 2, 1+\sqrt{2}, \ldots$ zdi se naraščajoče:

$$a_{n+1} \geq a_n \iff \sqrt{a_n} + \sqrt{a_{n-1}} \overset{?}{\geq} a_n \quad \Big\}^2$$

Očitno so vsi členi ≥ 0, zato smemo kvadrirati.

$$a_n + 2\sqrt{a_n}\sqrt{a_{n-1}} + a_{n-1} \overset{?}{\geq} a_n^2$$

$$2\sqrt{a_n a_{n-1}} \overset{?}{\geq} a_n^2 - a_n - a_{n-1} \quad \Big\}^2$$

$$4a_n a_{n-1} \overset{?}{\geq} a_n^4 + a_n^2 + a_{n-1}^2 - 2a_n^3 - 2a_n^2 a_{n-1} + 2a_n a_{n-1}$$

$$0 \overset{?}{\geq} a_n^4 + \ldots - 2a_n a_{n-1} \quad ?$$

Neenačbo $a_{n+1} \overset{\text{I.P.}}{\geq} a_n$ dokažimo z indukcijo!

① $n=1$: $a_2 = 1$, $a_1 = 0 \implies a_2 \geq a_1$ ✓

② I.K. $\quad a_{n+2} \geq a_{n+1} \iff a_{n+2} - a_{n+1} \geq 0$

$$a_{n+2} = \sqrt{a_{n+1}} + \sqrt{a_n}$$

$$a_{n+1} = \sqrt{a_n} + \sqrt{a_{n-1}}$$

$$a_{n+2} - a_{n+1} = \sqrt{a_{n+1}} + \cancel{\sqrt{a_n}} - (\cancel{\sqrt{a_n}} + \sqrt{a_{n-1}}) \implies$$

$$a_{n+2} - a_{n+1} = \sqrt{a_{n+1}} - \sqrt{a_{n-1}} \geq 0$$

če je $\sqrt{a_{n+1}} \geq \sqrt{a_{n-1}} \iff \underline{a_{n+1} \geq a_{n-1}}$

Po I.P. vemo $a_{n+1} \geq a_n$. Potrebovali bi še $a_n \geq a_{n-1}$.

RAZŠIRJENA INDUKCIJA RI:

Pri indukcijskem koraku privzamemo, da trditev drži za vse indekse $\leq n$.

<u>Naš primer</u>: Z uporabo RI lahko privzamemo $a_{n+1} \geq a_n$ in $a_n \geq a_{n-1}$

$\Longrightarrow a_{n+2} \geq a_{n+1}$ ✓

<u>Omejenost (navzgor)</u>: $a_n \leq M$?

$a_6 = \sqrt{1+\sqrt{2}} + \sqrt{2}$

- Predpostavimo, da je $\lim\limits_{n\to\infty} a_n = a$. Potem je $a = \sup\{a_n\}$ ← ZGORNJA MEJA

naraščajoča

$$a_{n+1} = \sqrt{a_n} + \sqrt{a_{n-1}}$$

$$a = \sqrt{a} + \sqrt{a} \iff a = 2\sqrt{a}$$

$a_1 \neq a$

$$a - 2\sqrt{a} = 0 \iff \sqrt{a}(\sqrt{a} - 2) = 0 \iff \cancel{a = 0} \text{ ali } \boxed{a = 4}$$

Dokažemo, da je $\underline{a_n \leq 4}$

① $n=1$: $a_1 = 0 \leq 4$ ✓

② I.K.: $a_{n+1} = \sqrt{a_n} + \sqrt{a_{n-1}}$

Po razširjeni indukciji lahko privzamemo, da $a_n \leq 4$ in $a_{n-1} \leq 4 \Longrightarrow$

$$a_{n+1} = \sqrt{a_n} + \sqrt{a_{n-1}} \leq \sqrt{4} + \sqrt{4} = 4$$ ✓

Povzetek: a_n naraščа, je navzgor omejena s 4 $\Rightarrow \exists \lim_{n\to\infty} a_n = a$

Že prej smo izračunali, da je $a=0$ ali $a=4$, ker $a=0$ ni možna limita.

Torej
$$\boxed{\lim_{n\to\infty} a_n = 4.}$$

---

11. Naj bo a poljubno pozitivno realno število. Dano je zaporedje

$$a_n = \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{\cdots}}}}}_{n}.$$

Določi a_1, a_2 in a_3. Dokaži, da zaporedje $\{a_n\}_{n\in\mathbb{N}}$ konvergira in izračunaj njegovo limito.

$a_1 = \sqrt{a}$, $a_2 = \sqrt{a+\sqrt{a}}$, $a_3 = \sqrt{a+\sqrt{a+\sqrt{a}}}$

$$\left.\begin{array}{l} \boxed{a_{n+1} = \sqrt{a+a_n}} \\ \boxed{a_1 = \sqrt{a}} \end{array}\right\}$$ Rekurzivno podano zaporedje.

$a>0 \Rightarrow a_n > 0$

Opazimo, da je $a_2 = \sqrt{a+\sqrt{a}} > \sqrt{a} = a_1 \Rightarrow$

$a_2 > a_1 \Rightarrow$ Domnevamo, da zaporedje raste:

$a_{n+1} > a_n$

① $n=1$: $a_2 > a_1$ ✓

② I.K.: $a_{n+2} \overset{?}{>} a_{n+1}$

$$\left.\begin{array}{l} a_{n+2} = \sqrt{a+a_{n+1}} \\ a_{n+1} = \sqrt{a+a_n} \end{array}\right\}$$ Po rekurziji.

$$\left.\sqrt{a+a_{n+1}} \overset{?}{\geq} \sqrt{a+a_n}\right\}^2$$

$$\cancel{a}+a_{n+1} \overset{?}{\geq} \cancel{a}+a_n \quad ✓ \quad \text{(po I.P.)}$$

POSPLOŠITEV: $\boxed{a_{n+1}=f(a_n)}$

Delamo z indukcijo:

$$\boxed{a_{n+1} \substack{\geq \\ \leq} a_n} \overset{?}{\Longrightarrow} \underbrace{a_{n+2} \substack{\geq \\ \leq} a_{n+1}}$$

$$\overset{?}{\Longrightarrow} \boxed{f(a_{n+1}) \substack{\geq \\ \leq} f(a_n)}$$

TA SKLEP VELJA, ČE JE f NARAŠČAJOČA:

$$x \leq y \Rightarrow f(x) \leq f(y)$$

Naš primer: $f(x)=\sqrt{a+x}$ je naraščajoča.

Ker je $a_2 \geq a_1 \Longrightarrow$ zaporedje raste.

Zaporedje je navzgor omejeno:

$$a_n \leq M$$

Izračunamo kandidate za limito. Limita (če obstaja) je (natančna) zgornja meja.

$$\begin{array}{ccc} a_{n+1} & = & \sqrt{a+a_n} \\ \downarrow & & \downarrow {\scriptstyle n\to\infty} \\ \alpha & = & \sqrt{a+\alpha} \end{array}$$

$$\alpha = \lim_{n\to\infty} a_n$$

POSPLOŠENO:

$$\left.\begin{array}{ccc} a_{n+1} & = & f(a_n) \\ \downarrow & & \downarrow \\ \alpha & = & f(\alpha) \end{array}\right\} f \text{ zvezna}$$

$$\Downarrow$$

$$\alpha^2 = a+\alpha$$

$$\alpha^2-\alpha-a=0$$

$$a_n>0 \Rightarrow \alpha \geq 0$$

$$\alpha_{1,2} = \frac{1\pm\sqrt{1+4a}}{2} = \begin{cases} \alpha = \dfrac{1-\sqrt{1+4a}}{2} < 0 \quad ✗ \\ \alpha = \dfrac{1+\sqrt{1+4a}}{2} \end{cases}$$

Če je a_n omejeno, velja $a_n \leq \frac{1+\sqrt{1+4a}}{2}$: I.P.

① $n=1$: $a_1 = \sqrt{a} \overset{?}{\leq} \underbrace{\frac{1+\sqrt{1+4a}}{2}}$

$\sqrt{1+4a} \geq \sqrt{4a} = 2\sqrt{a}$

$1+\sqrt{1+4a} \geq 1+2\sqrt{a} \geq 2\sqrt{a}$

$\frac{1+\sqrt{1+4a}}{2} \geq \frac{2\sqrt{a}}{2} = \sqrt{a}$ ✓

② I.K.: $a_{n+1} = \sqrt{a+a_n} \overset{?}{\leq} \frac{1+\sqrt{1+4a}}{2} \Big\}^2$

$\cancel{a}+a_n \overset{?}{\leq} \frac{1+2\sqrt{1+4a}+1+\cancel{4a}}{4}$

$4a_n \overset{?}{\leq} 2+2\sqrt{1+4a}$

$2a_n \overset{?}{\leq} 1+\sqrt{1+4a} \quad \Big\}:2$

$a_n \leq \frac{1+\sqrt{1+4a}}{2}$ ✓

Povzetek: Zaporedje a_n narašča in je navzgor omejeno, zato ima limito α. Izračunali smo, da je

$$\alpha = \frac{1+\sqrt{1+4a}}{2} = \lim_{n\to\infty} a_n$$

edina možna limita.

12. Izračunaj naslednji limiti.

(a) $\lim_{n\to\infty} \left(\frac{2n+3}{2n+5}\right)^{n+1}$,

(b) $\lim_{n\to\infty} \left(\frac{n^2+n-4}{n^2+n-3}\right)^{n^2+n+5}$.

(a) $\left(\frac{2n+3}{2n+5}\right)^{n+1}$, $n+1 \to \infty$, $\frac{2n+3}{2n+5} \xrightarrow{n\to\infty} 1$, ima limito tipa "$1^\infty$"

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2{,}7\ldots}$$

$$\boxed{\lim_{t\to 0}(1+t)^{1/t}=e}$$

$$\frac{2n+3}{2n+5}=\frac{2n+5-2}{2n+5}=1-\overset{t}{\frac{2}{2n+5}}$$

$$\left(\frac{2n+3}{2n+5}\right)^{n+1}=(1+t)^{\frac{1}{t}\cdot t\cdot(n+1)}=$$

$$=\Big(\underbrace{(1+t)^{\frac{1}{t}}}_{\to e}\Big)^{t\cdot(n+1)\to L}\Big\}\longrightarrow e^{L}$$

$$L=\lim_{n\to\infty}t\cdot(n+1)=\lim_{n\to\infty}\left(-\frac{2}{2n+5}\right)(n+1)=$$

$$=\lim_{n\to\infty}-\frac{2n+2}{2n+5}=-1$$

Torej: $\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right)^{n+1}=e^{-1}$

RECEPT: $\lim_{n\to\infty}a_n^{b_n}$, kjer je $\lim_{n\to\infty}a_n=1$ in $\lim_{n\to\infty}b_n=\pm\infty$.

$$a_n^{b_n}=(1+\underbrace{(a_n-1)}_{t})^{b_n}=(1+t)^{\frac{1}{t}\cdot t\cdot b_n}=$$

$$=\Big[\underbrace{(1+t)^{\frac{1}{t}}}_{\to e}\Big]^{\underset{\downarrow L}{t\cdot b_n}}\longrightarrow e^{L}$$

$$L=\lim_{n\to\infty}t\cdot b_n=\lim_{n\to\infty}(a_n-1)b_n$$

Velja: $\lim_{n\to\infty}a_n^{b_n}=e^{L}$, kjer je

$$L=\lim_{n\to\infty}b_n(a_n-1)$$

(b) $\lim_{n\to\infty}\left(\frac{n^2+n-4}{n^2+n-3}\right)^{n^2+n+5}$.

$$a_n = \frac{n^2+n-4}{n^2+n-3} \qquad b_n = n^2+n+5$$

$$L = \lim_{n\to\infty} b_n(a_n-1) = \lim_{n\to\infty}(n^2+n+5)\left(\frac{n^2+n-4}{n^2+n-3}-1\right) =$$

$$= \lim_{n\to\infty}(n^2+n+5)\cdot\frac{\cancel{n^2}+\cancel{n}-4-(\cancel{n^2}+\cancel{n}-3)}{n^2+n-3} =$$

$$= \lim_{n\to\infty}\frac{-n^2-n-5 \,:n^2}{n^2+n-3 \,:n^2} = -1$$

Torej: $\lim_{n\to\infty} a_n^{b_n} = e^L = e^{-1}$

13. Izračunaj limiti

$$\lim_{n\to\infty}\sqrt[n+1]{n} \qquad \text{in} \qquad \lim_{n\to\infty}\sqrt[n]{n+1}.$$

$$\boxed{\lim_{n\to\infty}\sqrt[n]{n} = 1} \qquad \sqrt[n]{n} = n^{\frac{1}{n}} \qquad \lim_{n\to\infty}\frac{n}{n+1} = 1$$

- $\sqrt[n+1]{n} = n^{\frac{1}{n+1}} = n^{\frac{1}{n}\cdot n\cdot\frac{1}{n+1}} = \left(\underbrace{n^{\frac{1}{n}}}_{\to 1}\right)^{\overbrace{n\cdot\frac{1}{n+1}}^{\to 1}} \longrightarrow 1$

VSILJEVANJE

$$\lim_{n\to\infty}\sqrt[n+1]{n} = 1$$

- $\sqrt[n]{n+1} = (n+1)^{\frac{1}{n}} = (n+1)^{\frac{1}{n+1}\cdot(n+1)\cdot\frac{1}{n}} =$

$$= \left(\underbrace{(n+1)^{\frac{1}{n+1}}}_{\to 1}\right)^{\overbrace{\frac{n+1}{n}}^{\to 1}} \longrightarrow 1$$

$$\lim_{n\to\infty}\sqrt[n]{n+1} = 1$$