ŠTEVILSKE VRSTE

1. Seštej vrsto $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Vrsta: $\sum_{n=1}^{\infty} a_n$, $a_n \in \mathbb{R}$

$\sum_{n=1}^{\infty} a_n$ konvergira, če konvergira zaporedje delnih vsot

$$\boxed{s_m = \sum_{n=1}^{m} a_n}.$$

Rečemo, da je $\lim_{m \to \infty} s_m$ je vsota vrste.

$$s_m = \sum_{n=1}^{m} a_n = a_1 + a_2 + \dots + a_m$$

Naš primer: $a_n = \frac{1}{n(n+1)}$

$$s_1 = a_1 = \frac{1}{2}$$

$$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$s_3 = \underbrace{a_1 + a_2} + a_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$$

$$s_4 = s_3 + a_4 = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$$

Domneva: $s_m = \frac{m}{m+1}$ I.P.

INDUKCIJA:

① $m = 1$: $s_1 = \frac{1}{2} = \frac{1}{1+1}$

② I.K.: $s_{m+1} = a_1 + a_2 + \dots + a_m + a_{m+1} =$

$$= s_m + a_{m+1} \underset{\text{I.P.}}{\overset{\uparrow}{=}} \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} =$$

$$= \frac{m(m+2)+1}{(m+1)(m+2)} = \frac{m^2+2m+1}{(m+1)(m+2)} = \frac{(m+1)^{\cancel{2}}}{\cancel{(m+1)}(m+2)} =$$

$$= \frac{m+1}{m+2} = \frac{m+1}{(m+1)+1} \quad ✓$$

Vsota vrste je $\lim\limits_{m\to\infty} s_m = \lim\limits_{m\to\infty} \frac{m}{m+1} = 1.$

2. Ugotovi konvergenco naslednjih vrst:

(a) $\sum_{n=1}^{\infty} \frac{n+1}{2n+1}$,

(b) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{(n+1)\sqrt{n}}$,

(a) Če vsota $\sum\limits_{n=1}^{\infty} a_n$ konvergira $\Rightarrow$ $\boxed{\lim\limits_{n\to\infty} a_n = 0}$

Naš primer: $a_n = \frac{n+1}{2n+1} \Rightarrow \lim\limits_{n\to\infty} a_n = \frac{1}{2} \neq 0$

$\Longrightarrow$ Vrsta ne konvergira (divergira).

(b) $a_n = \frac{\sqrt[3]{n}}{(n+1)\sqrt{n}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^{\frac{1}{3}} : n^{\frac{3}{2}}}{\underbrace{(n+1)\cdot n^{1/2}}_{n^{3/2}+n^{1/2}} : n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{\overset{\to 0}{n^{-\frac{7}{6}}}}{1 + \underset{\downarrow 0}{n^{-1}}} = 0$$

$$a_n = \frac{n^{-7/6}}{\underset{\text{za velike } n \ \nearrow \approx 1}{1+n^{-1}}} \approx n^{-7/6} = \frac{1}{n^{7/6}}$$

$$\boxed{\text{Vrsta } \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ konvergira} \Leftrightarrow s > 1}$$

Opomba: Za $s = 1$ dobimo $\sum\limits_{n=1}^{\infty} \frac{1}{n}$, tj.

harmonična vrsta, ki divergira, čeprav gredo členi proti 0.

Naš primer: $\sum_{n=1}^{\infty} \frac{1}{n^{7/6}}$ konvergira, ker $s = 7/6 > 1$.

PRIMERJALNI KRITERIJ: Opazujmo vrsto $\sum_{n=1}^{\infty} a_n$ z nenegativnimi členi, $a_n \geq 0$.

Recimo, da ocenimo $0 \leq c_n \leq a_n$ ali $a_n \leq b_n$.

1. Če vrsta $\sum_{n=1}^{\infty} c_n$ divergira $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ divergira.
2. Če vrsta $\sum_{n=1}^{\infty} b_n$ konvergira $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n$ konvergira.

Ugibamo (na osnovi $a_n \approx \frac{1}{n^{7/6}}$), da naša vrsta konvergira. Želimo oceno

$$a_n \leq b_n$$

$$a_n = \frac{n^{-7/6}}{\underset{1}{\underset{\vee}{1+n^{-1}}}} < \frac{n^{-7/6}}{1} = \frac{1}{n^{7/6}}$$

$$a_n < \frac{1}{n^{7/6}} = b_n$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{7/6}}$ konvergira $\overset{2}{\Longrightarrow}$

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ konvergira.

3. Ugotovi konvergenco naslednjih vrst:

(a) $\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt{2^n}}$,

(b) $\sum_{n=1}^{\infty} \frac{n^3}{e^n}$,

(c) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$,

(a) $a_n = \frac{2n-1}{\sqrt{2^n}}$

"RAČUNSKA" KRITERIJA ZA KONVERGENCO

- KVOCIENTNI KRITERIJ: $a_n > 0$

$D_n = \frac{a_{n+1}}{a_n}$, $D = \lim_{n\to\infty} D_n$ (predpostavimo, da ta limita obstaja)

1. $D < 1 \implies$ vrsta $\sum_{n=1}^{\infty} a_n$ konvergira
2. $D > 1 \implies$ —— " —— divergira

Če je $D = 1$, ne vemo...

- KORENSKI KRITERIJ: $a_n \geq 0$

$C_n = \sqrt[n]{a_n}$, $C = \lim_{n\to\infty} C_n$

1. $C < 1 \implies$ vrsta $\sum_{n=1}^{\infty} a_n$ konvergira
2. $C > 1 \implies$ —— " —— divergira

~~Naš primer~~: $a_n = \frac{2n-1}{2^{n/2}}$

$a_{n+1} = \frac{2(n+1)-1}{2^{(n+1)/2}}$

$$D_n = \frac{a_{n+1}}{a_n} = \frac{(2n+1)\,\cancel{2^{n/2}}}{(2n-1)\,\underbrace{2^{\cancel{n/2}+1/2}}_{2^{n/2}\cdot 2^{1/2}}} =$$

$$= \frac{(2n+1)}{(2n-1)\sqrt{2}}$$

$$\lim_{n\to\infty} D_n = \lim_{n\to\infty} \frac{2n+1 \;:n}{(2n-1)\sqrt{2}\;:n} = \frac{1}{\sqrt{2}} = D < 1$$

$\Longrightarrow$ vrsta konvergira

(b) $\sum_{n=1}^{\infty} \frac{n^3}{e^n}$, $a_n = \frac{n^3}{e^n}$

$$D_n = \frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{e^{n+1}} \frac{\cancel{e^n}}{n^3} = \frac{1}{e}\left(\underbrace{\frac{n+1}{n}}_{\to 1}\right)^3$$

$\Longrightarrow \lim_{n\to\infty} D_n = \frac{1}{e} = D < 1 \Longrightarrow$ vrsta konv.

Za primerjavo še korenski kriterij:

$$C_n = \sqrt[n]{a_n} = \sqrt[n]{\frac{n^3}{e^n}} = \frac{n^{3/n}}{e} \overset{*}{=} \frac{(\underbrace{\sqrt[n]{n}}_{\to 1})^3}{e}$$

$\lim_{n\to\infty} C_n = \frac{1}{e} < 1 \Longrightarrow$ konvergenca

Opomba: Za $a_n > 0$ velja

$$\boxed{C = \lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n = D},$$

torej sta kriterija enakovredna.

$* \quad n^{3/n} = \left(n^{\frac{1}{n}}\right)^3 = \left(\sqrt[n]{n}\right)^3$

(c) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$, $a_n = \frac{(n!)^2}{(2n)!}$

$$D_n = \frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2 \,(2n)!}{(2n+2)!\,(n!)^2} = \frac{((n+1)\cancel{n!})^2\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{(n!)^2}} =$$

$$= \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{(n+1)^2}{2\cancel{(n+1)}(2n+1)} = \frac{n+1}{2(2n+1)}$$

$\lim_{n\to\infty} D_n = \frac{1}{4} < 1 \Longrightarrow$ vrsta konvergira

4. Ugovoti konvergenco naslednjih vrst:

(a) $\sum_{n=1}^{\infty} \frac{3n}{(2-\frac{1}{n})^n}$,

(b) $\sum_{n=1}^{\infty} \left(\frac{n+4}{n}\right)^n$,

(a) $a_n = \frac{3n}{(2-\frac{1}{n})^n} > 0$

$c_n = \sqrt[n]{a_n} = \frac{\sqrt[n]{3n}}{2-\frac{1}{n}} = \frac{\sqrt[n]{3}\,\sqrt[n]{n}}{2-\frac{1}{n}} \Longrightarrow$ $\quad \sqrt[n]{3} \to 1,\ \sqrt[n]{n} \to 1,\ 2-\frac{1}{n} \to 2$

$\sqrt[n]{3} = 3^{\frac{1}{n}} \longrightarrow 3^0 = 1 \qquad \frac{1}{n} \to 0$

$\Longrightarrow \lim_{n\to\infty} c_n = \frac{1}{2} < 1 \Longrightarrow$ vrsta konv.

(b) $a_n = \left(\frac{n+4}{n}\right)^n$

$c_n = \sqrt[n]{a_n} = \frac{n+4}{n} \xrightarrow{n\to\infty} 1$??

$a_n = \left(1+\frac{4}{n}\right)^n = \left(1+\frac{4}{n}\right)^{\frac{n}{4}\cdot 4} = \Bigg(\underbrace{\left(1+\frac{4}{n}\right)^{\frac{n}{4}}}_{\downarrow\, e}\Bigg)^4$

$\Longrightarrow \lim_{n\to\infty} a_n = e^4 \neq 0 \Longrightarrow$

$\Longrightarrow$ vrsta divergira

Opomba: $\lim_{n\to\infty} a_n$ ni potrebno računati, ker $\frac{n+4}{n} > 1 \Rightarrow a_n = \left(\frac{n+4}{n}\right)^n > 1 \Longrightarrow$

$\Longrightarrow \lim_{n\to\infty} a_n \neq 0 \Longrightarrow$ divergenca

5. Določi vsa števila s, za katera naslednja vrsta konvergira.

$$\sum_{n=1}^{\infty} \left(\frac{n+s}{n+1}\right)^{n^2-n}.$$