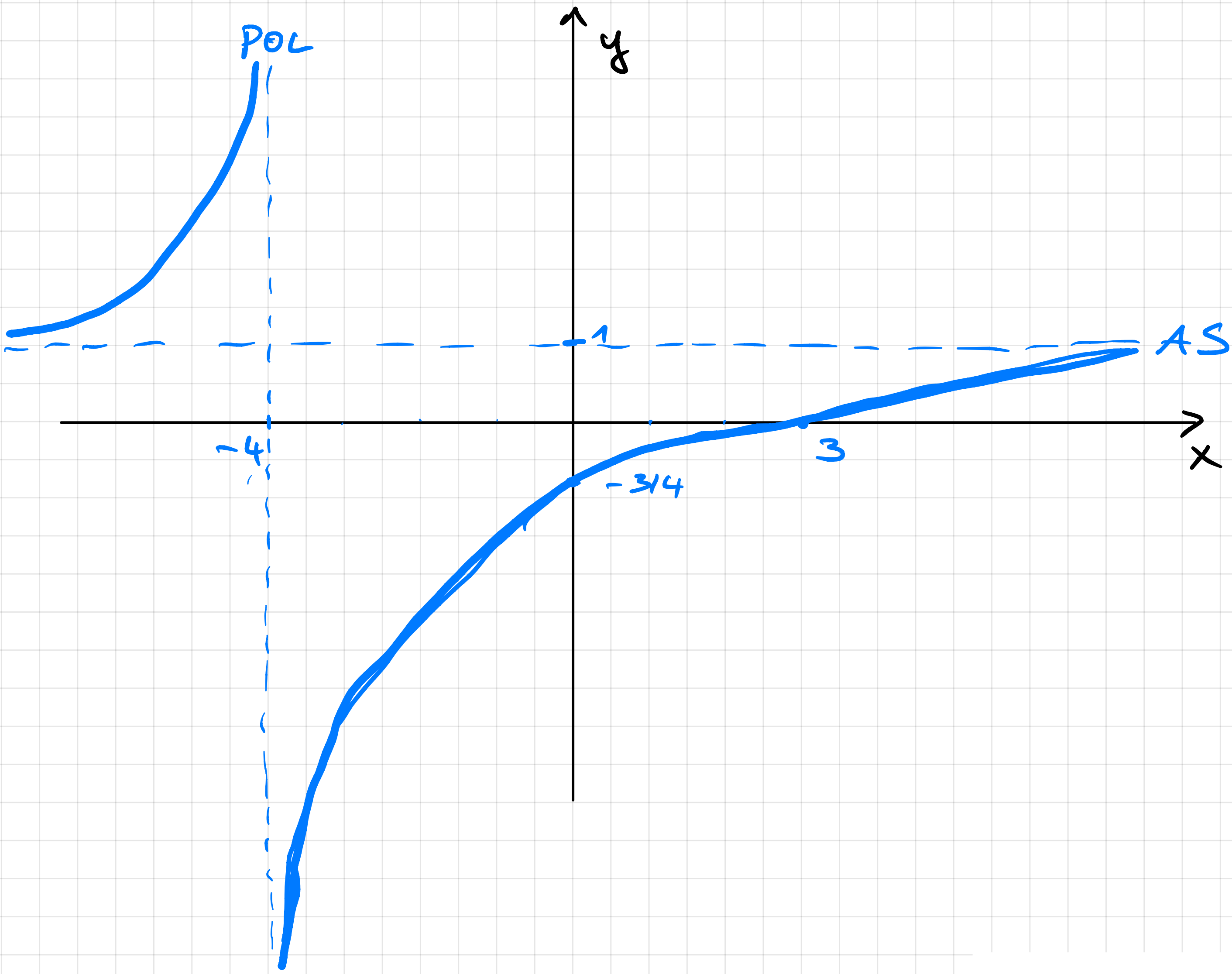


(b) $f(x) = \frac{x}{x^2+1}$,

- $D_f = \mathbb{R}$, ker $x^2+1 > 0$.
- Ničle: $x = 0$ prve stopnje
- Asimptota: Delimo števec z imenovalcem.

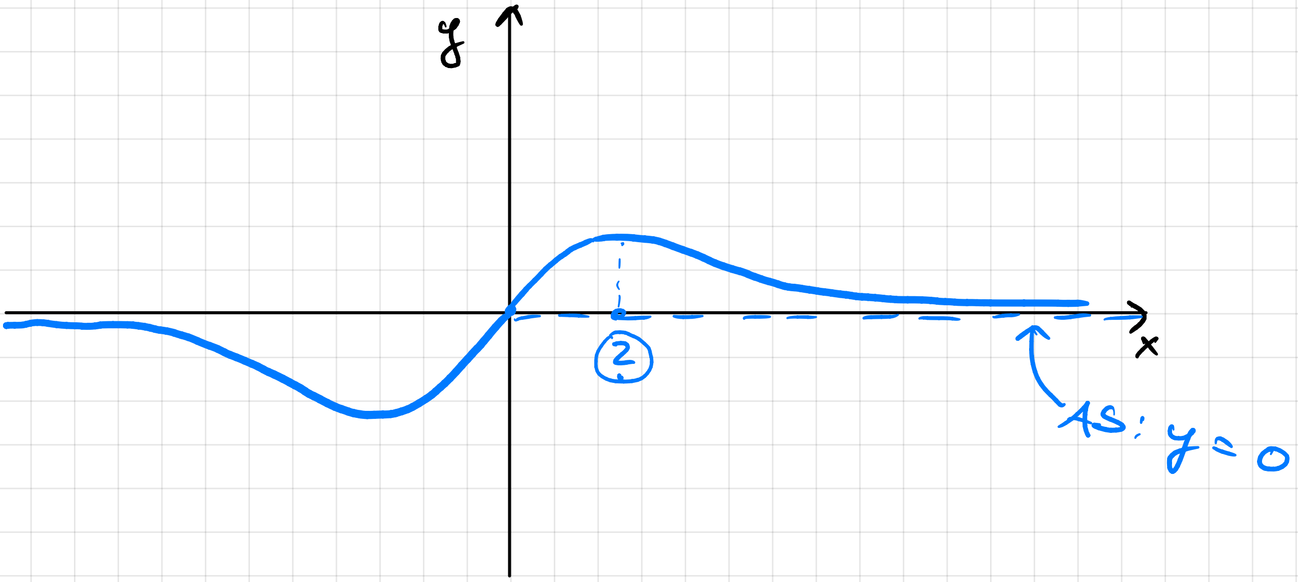
  $x : x^2+1 = 0$ (x ostane)

  AS: $y = 0$

- f je liha: $f(-x) = \frac{-x}{(-x)^2+1} = -\frac{x}{x^2+1} = -f(x)$

  $\Longrightarrow$ Graf je simetričen čez izhodišče.

- $f(0) = 0$



(c) $f(x) = \frac{x^2+2x-15}{x^2+5x+6}$,

- Poli: $x^2+5x+6=0 \iff (x+3)(x+2)=0$
  $x=-3,\ x=-2$ pola prve stopnje $\Big\} \Rightarrow D_f = \mathbb{R} \setminus \{-3,-2\}$
- Ničle: $x^2+2x-15=0 \iff (x+5)(x-3)=0$
  $x=-5,\ x=3$ ničli prve stopnje
- $f(0) = -\frac{15}{6} = -\frac{5}{2} = -2{,}5$
- AS: $x^2+2x-15 : x^2+5x+6 = 1$

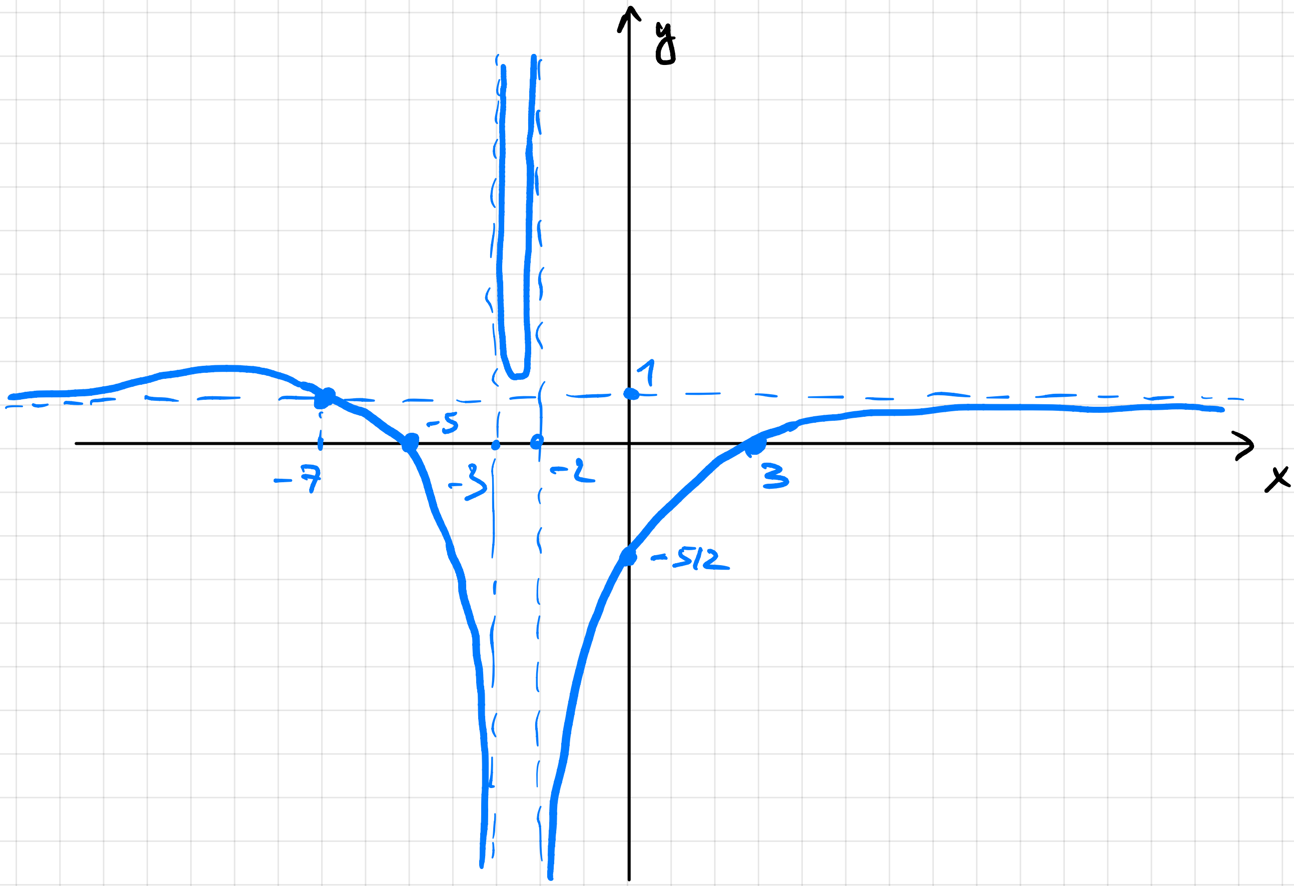
$$\begin{array}{l} x^2+2x-15 \\ -x^2 \mp 5x \mp 6 \\ \hline \quad -3x-21 \end{array}$$

$-3x-21$ ostanek

$$f(x) = 1 - \frac{3x+21}{x^2+5x+6}$$

AS: $y=1$, graf seka asimptoto, ko je ostanek $=0 \iff 3x+21=0 \iff x=-7$

Oba pola sta lihe stopnje $\Rightarrow$ graf gre z ene strani v ∞, z druge pa v $-\infty$.



DEF • f je <u>soda</u>, če $f(-x) = f(x)$ (graf simetričen čez os y)

• f je <u>liha</u>, če $f(-x) = -f(x)$ (simetrična čez $(0,0)$)

(d) $f(x) = \frac{x^3+1}{x^2+x-2}$.

• Poli: $x^2+x-2 = 0 \iff (x+2)(x-1) = 0 \implies$

$x=-2$, $x=1$ pola 1. stop. ($D_f = \mathbb{R} \setminus \{-2, 1\}$)

• Ničle: $x^3+1 = 0 \iff x^3 = -1 \iff x = -1$

$x^3+1 = \underline{(x+1)}^{1}(x^2-x+1)$ → $D = 1-4<0$ (ni $\mathbb{R}$-rešitev.)

$a^3+b^3 = (a+b)(a^2-ab+b^2)$

$x=-1$ ničla 1. stop.

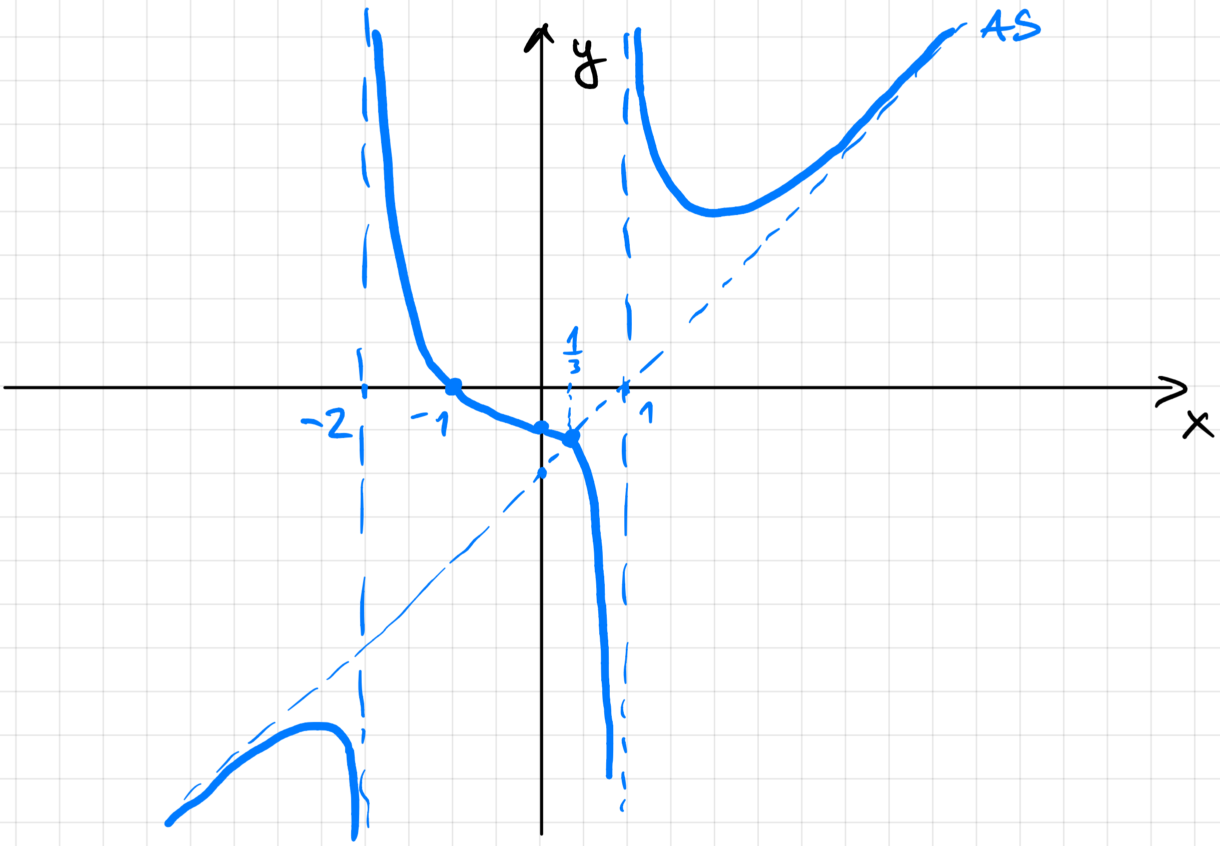
• $f(0) = -\frac{1}{2}$

- AS: $x^3+1 : x^2+x-2 = x-1$

$$-x^3 - x^2 + 2x$$
$$-x^2+2x+1$$
$$+x^2+x-2$$
$$3x-1 \text{ ost.}$$

AS: $y = x-1$

Graf seka AS, če $3x-1=0 \Rightarrow$ pri $x=\frac{1}{3}$



8. Reši naslednji trigonometrični enačbi

(a) $2\sin^2 x - \sin x = 1$,

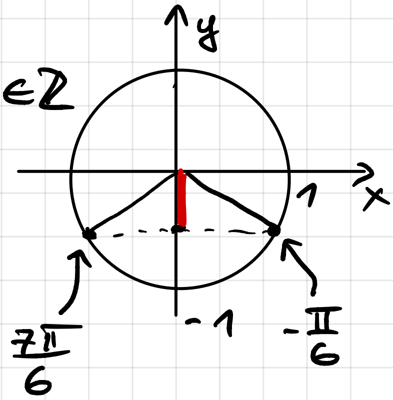
(b) $4\sin^2 x - \sin^2(2x) = 1$.

(a) $t = \sin x$; $2t^2 - t - 1 = 0$

$$t_{1,2} = \frac{1 \pm \sqrt{9}}{4} = \frac{1 \pm 3}{4} = \begin{cases} 1 \\ -1/2 \end{cases}$$

① $\sin x = 1 \iff x = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

② $\sin x = -\frac{1}{2} \iff \left.\begin{array}{l} x = \frac{7\pi}{6} + 2k\pi \\ x = -\frac{\pi}{6} + 2k\pi \end{array}\right\} k \in \mathbb{Z}$



(b) $\sin 2x = 2 \sin x \cos x$

$$4\sin^2 x - (2\sin x \cos x)^2 = 1$$

$$4\sin^2 x - 4\sin^2 x \cos^2 x = 1$$

$$4\sin^2 x \underbrace{(1 - \cos^2 x)}_{\sin^2 x} = 1$$

$$4\sin^4 x = 1 \iff \sin^4 x = \frac{1}{4} \iff$$

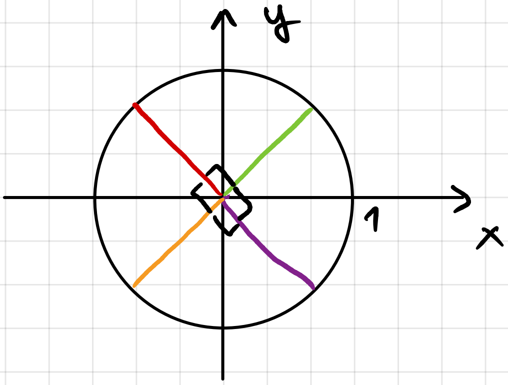
$$\iff \sin x = \pm \sqrt[4]{\frac{1}{4}} = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$$

$\sqrt[4]{4} = \sqrt{2}$

① $\sin x = \frac{\sqrt{2}}{2} \iff \left.\begin{array}{l} x = \frac{\pi}{4} + 2k\pi \\ x = \frac{3\pi}{4} + 2k\pi \end{array}\right\} k \in \mathbb{Z}$

$\pi - \frac{\pi}{4}$

② $\sin x = -\frac{\sqrt{2}}{2} \iff \left.\begin{array}{l} x = -\frac{\pi}{4} + 2k\pi \\ x = -\frac{3\pi}{4} + 2k\pi \end{array}\right\} k \in \mathbb{Z}$



Rešitve poenotimo:

$$\boxed{x = \frac{\pi}{4} + k \cdot \frac{\pi}{2}, \quad k \in \mathbb{Z}}$$

11. Reši naslednje enačbe.

(a) $\frac{1}{\ln x} = 2 + \ln\left(\frac{1}{x}\right)$,

(b) $4^{\sqrt{x}} + 4 = 5 \cdot 2^{\sqrt{x}}$,

(a) $\ln\left(\frac{1}{x}\right) = \ln\left(x^{-1}\right) = -\ln x$

$\boxed{\log_a b^c = c \log_a b,\ b>0}$

$\frac{1}{\ln x} = 2 - \ln x \ \Big\}\cdot \ln x \iff 1 = 2\ln x - \ln^2 x$

$t = \ln x:\ 1 = 2t - t^2 \iff t^2 - 2t + 1 = 0$

$(t-1)^2 = 0 \iff t = 1 = \ln x \iff \underline{x = e}$

(b) $4^{\sqrt{x}} = (2^2)^{\sqrt{x}} = 2^{2\sqrt{x}}$

$2^{2\sqrt{x}} + 4 = 5 \cdot 2^{\sqrt{x}} \iff 2^{2\sqrt{x}} - 5 \cdot 2^{\sqrt{x}} + 4 = 0 \iff$

$\iff 2^{\sqrt{x}}\left(2^{\sqrt{x}} - 5\right) + 4 = 0$

$t = 2^{\sqrt{x}}:\quad t(t-5) + 4 = 0$

$t^2 - 5t + 4 = 0$

$(t-1)(t-4) = 0$

1 $t = 1 = 2^{\sqrt{x}} \implies \sqrt{x} = 0 \implies \underline{x = 0}$

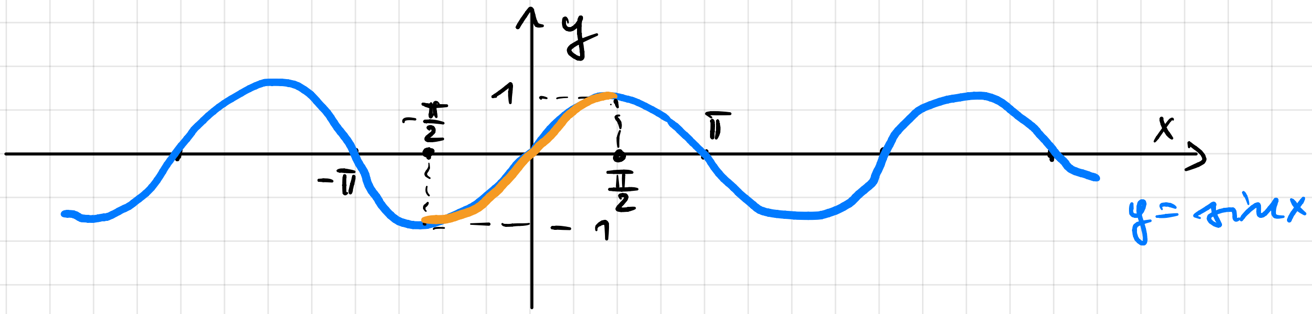
2 $t = 4 = 2^{\sqrt{x}} \implies \sqrt{x} = 2 \implies \underline{x = 4}$

12. Določi definicijsko območje funkcije

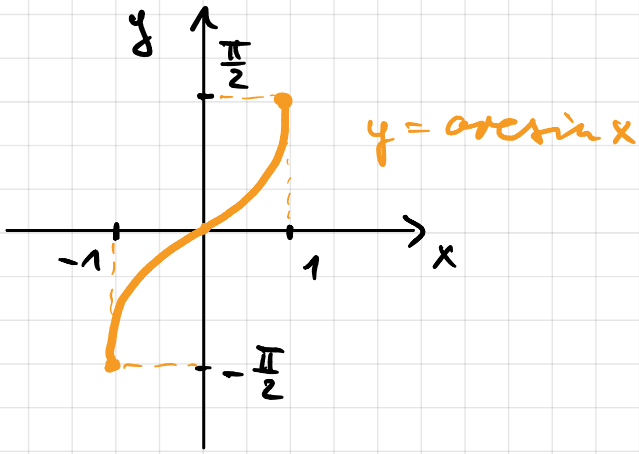
$$f(x) = \arcsin\left(\frac{x-1}{2}\right)$$

in skiciraj njen graf.

arcsin: inverzna funkcija od sin



Funkcijo sin omejimo na $[-\frac{\pi}{2}, \frac{\pi}{2}]$ in dobimo bijekcijo $\underline{\sin}: [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$. Inverzna funkcija je $\arcsin: [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$:

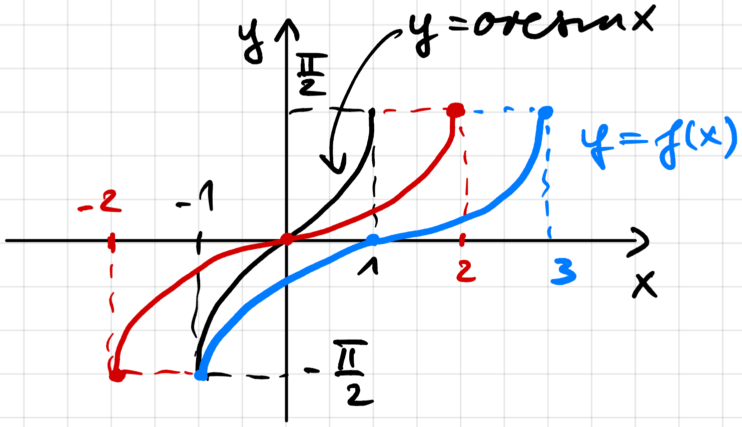


$D_{\arcsin} = [-1, 1] \Rightarrow D_f: \frac{x-1}{2} \in [-1, 1] \overset{/\cdot 2}{\Longleftrightarrow}$

$\Longleftrightarrow x - 1 \in [-2, 2] \underset{/+1}{\Longleftrightarrow} x \in [-1, 3] \Longrightarrow$

$\Longrightarrow \underline{D_f = [-1, 3]}$

1) $g(x) = \arcsin \frac{x}{2}$ ... raztег za faktor 2 v smeri x-osi:



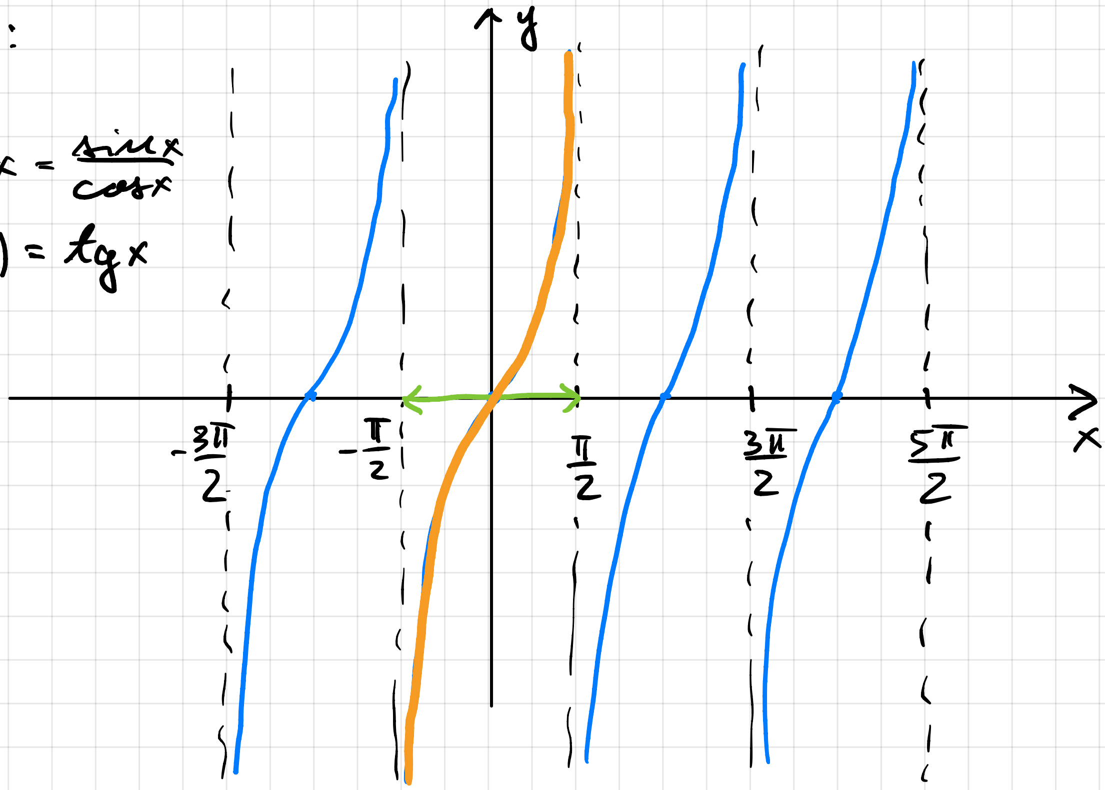
2) $f(x) = g(x-1)$ ---- premik za 1 v smeri osi x. $\left[f(x) = \arcsin \frac{x-1}{2}\right]$

13. Skiciraj graf funkcije
$$f(x) = 2\text{arctg}(x-1) + 1.$$

arctg:

$y = \text{tg}\, x = \frac{\sin x}{\cos x}$

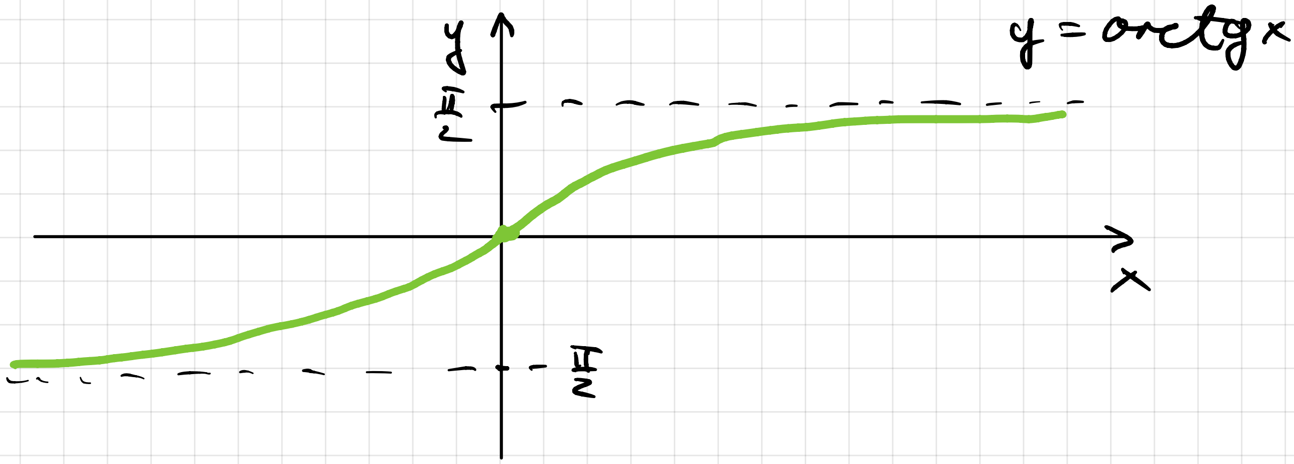
$\text{tg}(x+\pi) = \text{tg}\, x$



tg je periodična, zato ni injektivna.

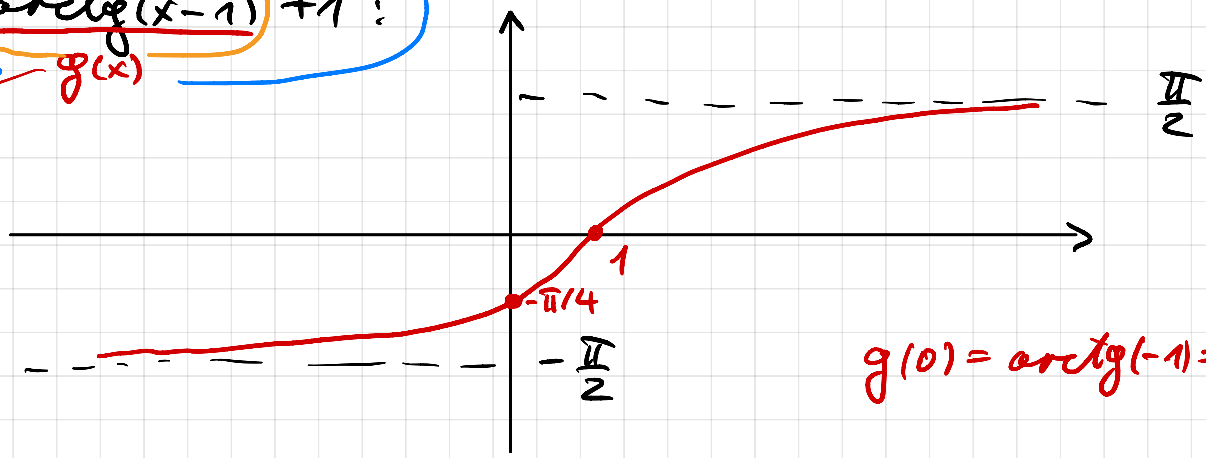
Izberemo vejo: $\underbrace{\text{tg}: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}}_{\text{bijektivna}}$ in njen

inverz je $\text{arctg}: \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



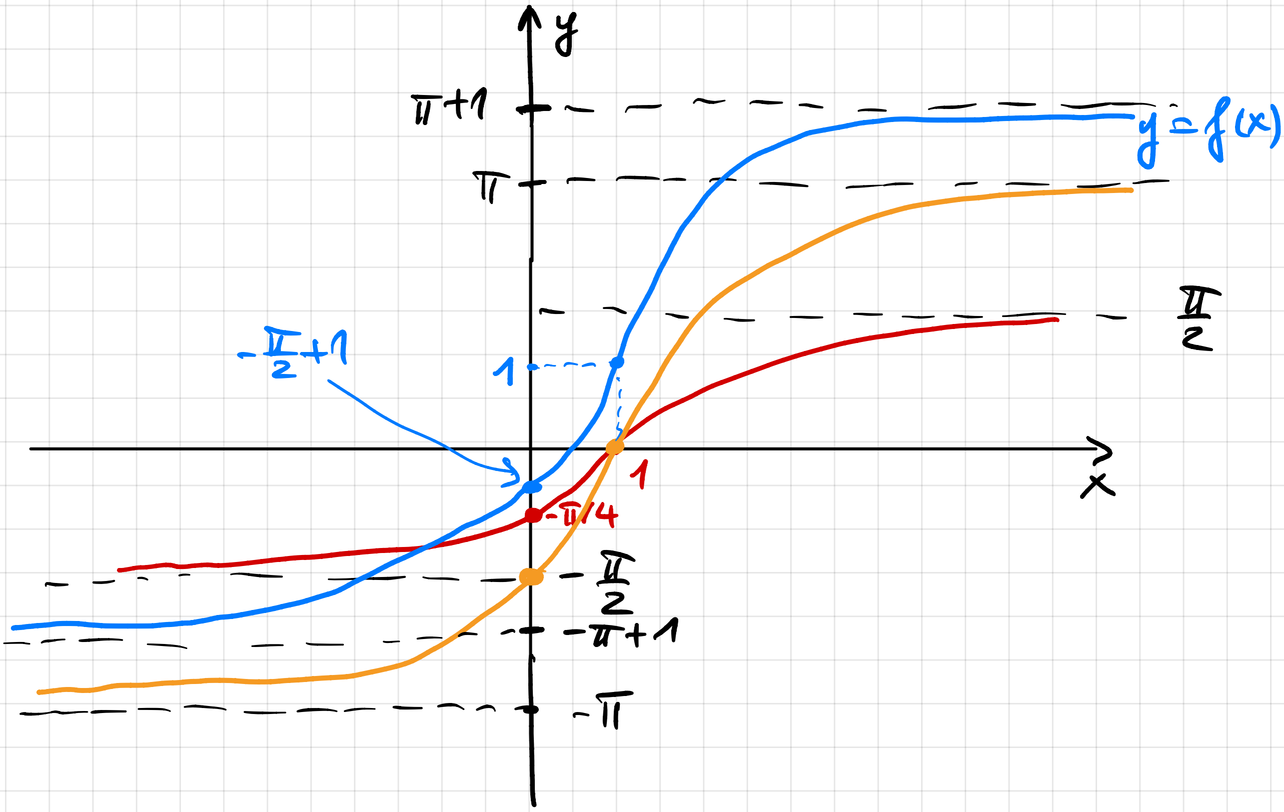
$f(x) = \underbrace{\overbrace{2\underline{\operatorname{arctg}(x-1)}}^{h(x)}}_{g(x)} + 1$:

premik za 1 v desno



$g(0) = \operatorname{arctg}(-1) = -\frac{\pi}{4}$

$h(x) = 2g(x)$ ... razteg za 2 v smeri y.



$f(x) = h(x) + 1$ ... premik za 1 v smer y

14. Dokaži, da je funkcija

$$f(x) = \ln(x + \sqrt{x^2+1})$$

liha.

f je liha, če je $f(-x) = -f(x)$.

① D_f: $x + \sqrt{x^2+1} > 0 \iff \sqrt{x^2+1} > -x$

$\sqrt{x^2+1} > \sqrt{x^2} = |x| \geq -x \implies$

$\implies \sqrt{x^2+1} > -x \implies D_f = \mathbb{R}$

② $f(-x) = -f(x)$

$$f(-x) = \ln(-x+\sqrt{(-x)^2+1}) = \ln(-x+\sqrt{x^2+1}) \stackrel{?}{=}$$
$$= -f(x) = -\ln(x+\sqrt{x^2+1})$$

$$\ln(-x+\sqrt{x^2+1}) \stackrel{?}{=} -\ln(x+\sqrt{x^2+1}) \Big\} e^{\cdots}$$

$$-x+\sqrt{x^2+1} \stackrel{?}{=} \frac{1}{x+\sqrt{x^2+1}}$$ odpravimo ulomke

$$(a = b \iff e^a = e^b)$$

$$(-x+\sqrt{x^2+1})(x+\sqrt{x^2+1}) \stackrel{?}{=} 1$$

$$(\sqrt{x^2+1})^2 - x^2 = \cancel{x^2}+1-\cancel{x^2} = 1 \checkmark$$

15. Dokaži, da velja

$$\operatorname{arctg}\left(\frac{1}{2}\right) + \operatorname{arctg}\left(\frac{1}{3}\right) \stackrel{?}{=} \frac{\pi}{4}.$$

Na enačbi uporabimo tg:

$$\boxed{\operatorname{tg}\left(\operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{3}\right) \stackrel{?}{=} \operatorname{tg}\frac{\pi}{4}}$$

$$\boxed{\operatorname{tg}(\alpha+\beta) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1-\operatorname{tg}\alpha\operatorname{tg}\beta}}$$

$$\frac{\operatorname{tg}(\operatorname{arctg}\frac{1}{2}) + \operatorname{tg}(\operatorname{arctg}\frac{1}{3})}{1-\operatorname{tg}(\operatorname{arctg}\frac{1}{2})\operatorname{tg}(\operatorname{arctg}\frac{1}{3})} \stackrel{?}{=} 1$$

$$\operatorname{tg}(\operatorname{arctg} x) = x \qquad \forall x \in \mathbb{R}$$
$$\operatorname{arctg}(\operatorname{tg} x) = x \qquad \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \,!$$

$$\frac{(\frac{1}{2}+\frac{1}{3})\cdot 6}{(1-\frac{1}{2}\cdot\frac{1}{3})\cdot 6} = \frac{3+2}{6-1} = \frac{5}{5} \stackrel{?}{=} 1 \checkmark$$

$\operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{3} = \gamma$

Ugotovili smo $\boxed{\operatorname{tg}\gamma = \operatorname{tg}\frac{\pi}{4}} \overset{?}{\Longrightarrow} \gamma = \frac{\pi}{4}$

Iz $\operatorname{tg}\gamma = \operatorname{tg}\frac{\pi}{4}$ lahko sklepamo, da je

$$\gamma = \frac{\pi}{4} + k\pi \quad (\pi \text{ perioda za } \operatorname{tg}) \qquad k \in \mathbb{Z}$$

$\operatorname{arctg}\frac{1}{2} \in (0, \frac{\pi}{2})$ in $\operatorname{arctg}\frac{1}{3} \in (0, \frac{\pi}{2}) \Longrightarrow$

$\Longrightarrow \underbrace{\operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{3}}_{\gamma} \in (0, \pi)$

$$\underbrace{\gamma \in (0,\pi) \text{ in } \gamma = \frac{\pi}{4} + k\pi}$$

$$\Downarrow$$

$$\boxed{\gamma = \frac{\pi}{4}}$$

5. Reši naslednje enačbe.

(a) $\cos x - \operatorname{tg}^2 x = 1$,

(b) $7 + \cos(10x) = 8\cos(5x)$,

~~(c) $64^x - 3\,16^x + 4 + 3^x = 0$,~~

(d) $\ln x \ln(\frac{x}{e^5}) = -6$.

(a) $\cos x - \frac{\sin^2 x}{\cos^2 x} = 1 \quad / \cdot \cos^2 x$

$\cos^3 x - \sin^2 x = \cos^2 x$

$\sin^2 x = (1 - \cos^2 x)$

$\cos^3 x - 1 + \cancel{\cos^2 x} = \cancel{\cos^2 x} \Longleftrightarrow \cos^3 x = 1 \Longleftrightarrow \cos x = 1$

$\Longleftrightarrow x = 0 + 2k\pi, k \in \mathbb{Z}$

$$\underline{x = 2k\pi, \ k \in \mathbb{Z}}$$

(b) $7+\cos(2t) = 8\cos t$ $\qquad \boxed{\cos(2t) = \cos^2 t - \sin^2 t}$

$5x = t$

$$7 + \cos^2 t - \underbrace{\sin^2 t}_{1-\cos^2 t} = 8\cos t$$

$$7 + 2\cos^2 t - 1 = 8\underbrace{\cos t}_{u} \iff 2u^2 - 8u + 6 = 0$$

$$u^2 - 4u + 3 = 0 \iff (u-3)(u-1) = 0$$

① $u = 3 = \cos t$, ne dobimo rešitve

② $u = 1 = \cos 5x \iff 5x = 2k\pi,\ k \in \mathbb{Z}$

$$x = \frac{2k\pi}{5},\ k \in \mathbb{Z}$$

(d) $\ln x \cdot \ln\left(\frac{x}{e^5}\right) = -6$

$$\ln x(\ln x - \ln e^5) = -6$$

$$\ln x(\ln x - 5) = -6$$

$t = \ln x$: $\quad t^2 - 5t + 6 = 0$

$$(t-3)(t-2) = 0$$

① $t = 3 = \ln x \implies x = e^3$

② $t = 2 = \ln x \implies x = e^2$

6. Skiciraj grafe.

(a) $f(x) = 8\ln(\sqrt[4]{x+1}) - 1$

$D_f: (\sqrt[4]{x+1} > 0 \iff x > -1) \implies D_f = (-1, \infty)$

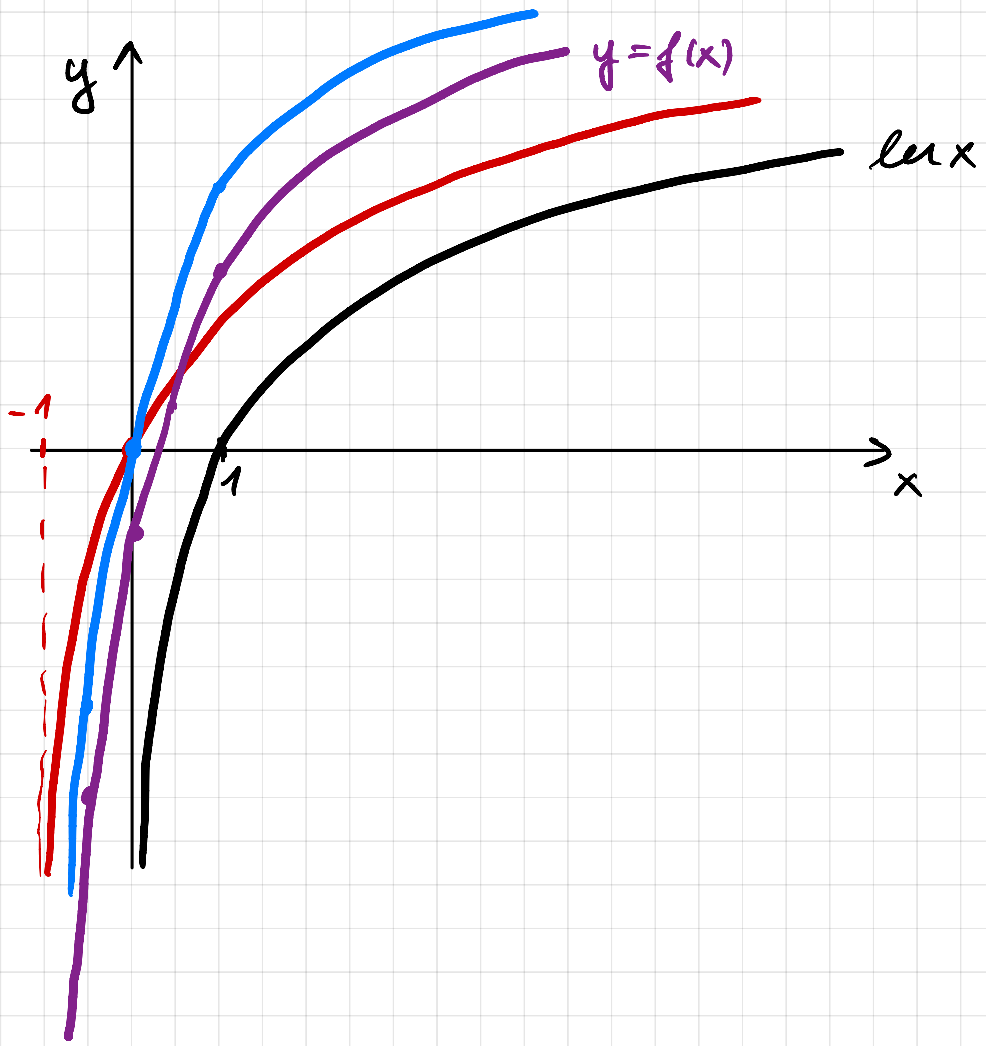
$$f(x) = 8\ln((x+1)^{1/4}) - 1 = 2\ln(x+1) - 1$$

Izhodiščni graf: $y = \ln x$

$y = \ln(x+1)$ ... premik za 1 v levo

$y = 2\ln(x+1)$ ... razteg za 2 v y-smeri

$y = 2\ln(x+1) - 1$ -- premik za 1 dol



(k) $f(x) = 2\sin x - 2\cos x + 1$ (Nasvet: izraz $\sin x - \cos x$ zapiši v obliki $b\sin(x+a)$ za primerna a in b,

$f(x) = 2(\sin x - \cos x) + 1$

$$\boxed{\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\sin x - \cos x = \cos\left(\frac{\pi}{2} - x\right) - \cos x = -2\sin\frac{\pi}{4}\sin\left(\frac{\frac{\pi}{2}-2x}{2}\right) =$$

$\frac{\pi}{4} - x$

$$\boxed{\begin{aligned} \sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha \\ \cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha \end{aligned}}$$

$$= -2\cdot\frac{\sqrt{2}}{2}\left(-\sin\left(x-\frac{\pi}{4}\right)\right) = \sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$$

$\sin(-\alpha) = -\sin\alpha$

$$f(x) = 2\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1$$

Izhodiščni graf: $y = \sin x$

$y = \sin(x - \frac{\pi}{4})$ ... premik za $\frac{\pi}{4}$ desno

$y = 2\sqrt{2}\sin(x - \frac{\pi}{4})$ ... razteg za $2\sqrt{2}$ navpično

$y = 2\sqrt{2}\sin(x - \frac{\pi}{4}) + 1 = f(x)$ ... premik za 1 gor