LIMITE IN ZVEZNOST FUNKCIJ

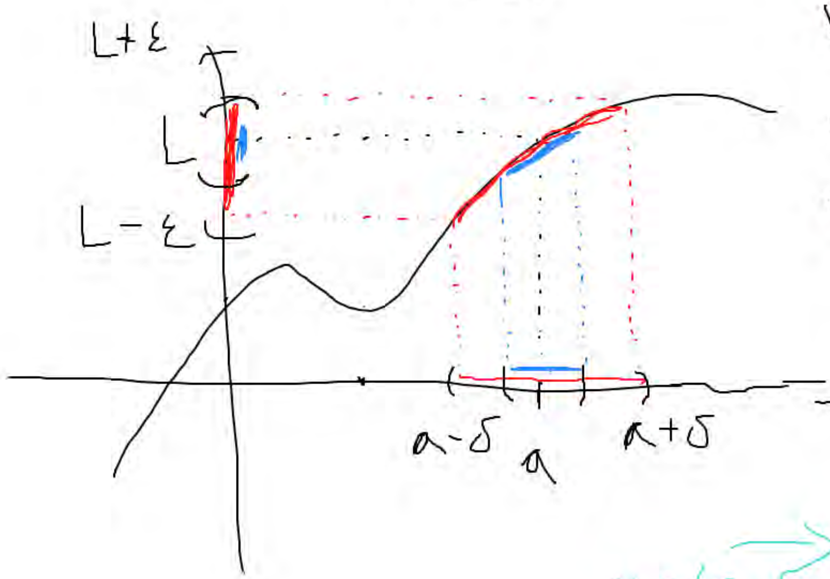
$\lim\limits_{x\to a} f(x) = L \overset{DEF}{\iff} \forall \varepsilon > 0\ \exists \delta > 0\ \forall x:$ V KOLOKVIJ 1.3 ?

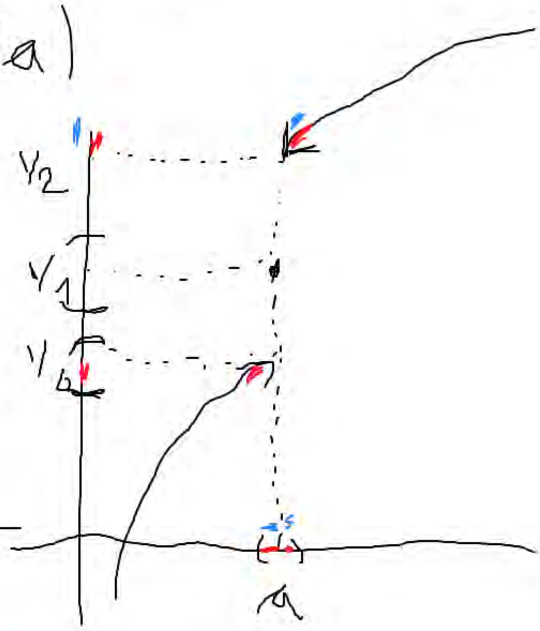
$$\underbrace{|x-a|<\delta}_{x\in(a-\delta,\,a+\delta)} \Rightarrow \underbrace{|f(x)-L|<\varepsilon}_{f(x)\in(L-\varepsilon,\,L+\varepsilon)}$$

f zvezna v a $\overset{DEF}{\iff} \forall \varepsilon > 0\ \exists \delta > 0\ \forall x:$

$$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$$

f ZVZ v a $\iff \lim\limits_{x\to a} f(x) = f(a)$

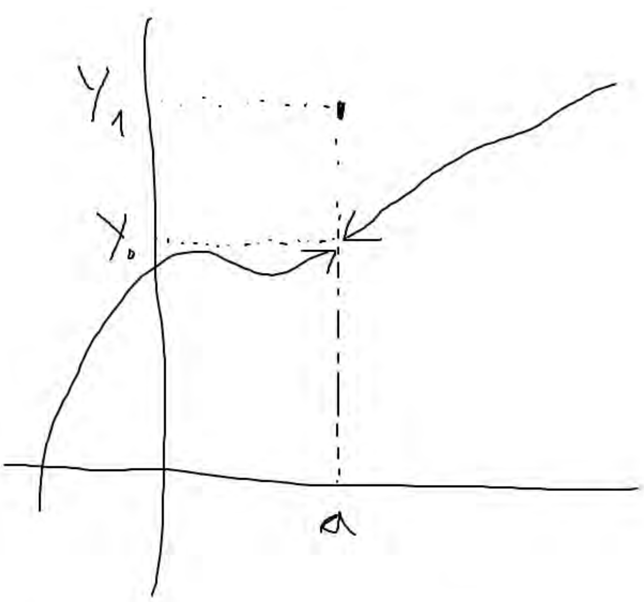




$x\in(a-\delta,a) \rightarrow Y_0 = \lim\limits_{x\uparrow a} f(x)$

$Y_1 = f(a)$

$x\in(a,a+\delta) \rightarrow Y_2 = \lim\limits_{x\downarrow a} f(x)$



$$\left.\begin{array}{l} Y_0 = \lim\limits_{x\to a} f(x) \\ Y_1 = f(a) \end{array}\right\} \Rightarrow f \text{ ni zvezna v } a$$

(1) $f(x) = x^2$, $\lim\limits_{x \to 1} f(x) = 1$

Za vsak $\varepsilon > 0$ moramo najti $\delta > 0$ (odvisen od ε), da iz $|x-1| < \delta$ sledi $|x^2 - 1| \overset{?}{<} \varepsilon$.

Naj velja $|x-1| < \delta$. Radi bi ocenili $|x^2-1|$.

Velja $|x^2-1| = |(x-1)(x+1)| = |x-1| \cdot |x+1| \leq \delta |x+1|$

$$= \delta \cdot |(x-1)+2| \leq \delta (|x-1| + |2|) \leq \delta(\delta+2) \leq$$

$$\leq \delta(1+2) = 3\delta \overset{?}{\leq} \varepsilon$$

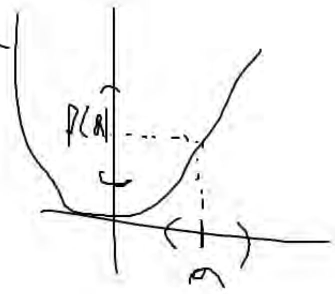
velja le, če je $\delta < 1$

Torej vzamemo $\delta := \min\{\frac{\varepsilon}{3}, 1\}$.

S tem je zagotovljeno, da vse neenakosti držijo in je dokaz končan.

3. $f(x)=x^2$ zvezna na $\mathbb{R}$

za vsak $a\in\mathbb{R}$ in za vsak $\varepsilon>0$ moramo najti $\delta>0$, da velja



$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon$

oz. $|x-a|<\delta \Rightarrow |x^2-a^2|<\varepsilon$

Naj bo $\varepsilon>0$ podan. Iščemo δ.

$$|x^2-a^2| = |x-a|\,|x+a| \leq \delta\,|x+a| = \delta\,|(x-a)+2a| \leq$$

$$\leq \delta\,(|x-a|+|2a|) \leq \delta(\delta+2|a|) \leq \delta(\underset{C>0}{1}+2|a|) \leq \varepsilon$$

če $\delta\leq 1$

Torej vzamemo $\delta := \min\left\{\frac{\varepsilon}{1+2|a|}, 1\right\}$

$\min\left\{\frac{\varepsilon}{C+2|a|}, C\right\}$ $\square$

2 (a) $\lim\limits_{x\to 1}\frac{x^2-3}{2x^2+2x-1} = \frac{1^2-3}{2\cdot 1^2+2\cdot 1-1} = \frac{-2}{3}$

(b) $\lim\limits_{x\to 1}\frac{x^2-1}{2x^2-x-1} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 1}\frac{\cancel{(x-1)}(x+1)}{(2x+1)\cancel{(x-1)}} = \frac{2}{3}$

ALTERNATIVNO: $\lim\limits_{x\to 1}\frac{2x-0}{2\cdot 2x-1-0} = \frac{2}{3}$

(c) $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a} \overset{\frac{0}{0}}{=} \lim\limits_{x\to a}\frac{\cancel{(x-a)}(x^{n-1}+x^{n-2}a+x^{n-3}a^2+x^{n-4}a^3+\dots+a^{n-1})}{\cancel{x-a}}$

$= a^{n-1}+a^{n-1}+\dots+a^{n-1} = na^{n-1}$

ALTERNATIVNO: $\lim\limits_{x\to a}\frac{nx^{n-1}-0}{1-0} = na^{n-1}$

(d) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\frac{\overset{a}{\sqrt{1+x}}-\overset{b}{\sqrt{1-x}}}{x}\cdot\frac{\overset{a}{\sqrt{1+x}}+\overset{b}{\sqrt{1-x}}}{\sqrt{1+x}+\sqrt{1-x}}$

$= \lim\limits_{x\to 0}\frac{\sqrt{1+x}^2-\sqrt{1-x}^2}{x(\sqrt{1+x}+\sqrt{1-x})} =$

$= \lim\limits_{x\to 0}\frac{(\cancel{1}+x)-(\cancel{1}-x)}{x(\sqrt{1+x}+\sqrt{1-x})} =$

$= \lim\limits_{x\to 0}\frac{2\cancel{x}}{\cancel{x}(\sqrt{1+x}+\sqrt{1-x})} = \frac{2}{\sqrt{1}+\sqrt{1}} = 1$

$(a-b)\cdot(a+b) = a^2-b^2$

ODVODI

$\boxed{x^{a\,\prime} = a x^{a-1}}$ $\quad \sin x' = \cos x$ $\quad \cos x' = -\sin x$

$\ln x' = \frac{1}{x}$, $\quad c' = 0$

$e^{x\,\prime} = e^x$

L'HOSPITAL:

če $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ ali $\pm\infty$,

in če na $(a-\delta, a+\delta)$ velja $g'(x)\neq 0$,

potem

$\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

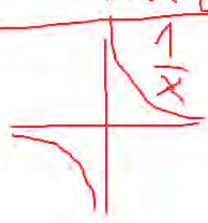
NEDOLOČENI IZRAZI: $\frac{0}{0}, 0\cdot\infty, \frac{\infty}{\infty}, 1^{\infty}, \infty-\infty$

$\left.\begin{array}{l} \frac{x}{x^2}\xrightarrow{x\to\infty} 0 \\ \frac{x}{x}\xrightarrow{x\to\infty} 1 \\ \frac{x^2}{x}\xrightarrow{x\to\infty}\infty \end{array}\right\} \frac{\infty}{\infty}$

DOLOČENI IZRAZI:

$c^{\infty} = \begin{cases}\infty; & \text{če } c>1 \\ 0; & \text{če } 0\leq c<1\end{cases}$, $\quad c\cdot\infty = \begin{cases}\infty; & c>0 \\ -\infty; & c<0\end{cases}$

$\frac{c}{\infty} = 0$, $\quad \frac{1}{+0} = +\infty$, $\frac{1}{-0} = -\infty$



(e) $\lim\limits_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1} \overset{\frac{0}{0}}{=} \lim \frac{\sqrt[3]{x}-1}{x-1} \cdot \frac{(x^{2/3}+x^{1/3}+1)}{(x^{2/3}+x^{1/3}+1)} =$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$= \lim\limits_{x\to 1} \frac{\cancel{\sqrt[3]{x}^3 - 1^3}}{\cancel{(x-1)}(x^{2/3}+x^{1/3}+1)} = \lim\limits_{x\to 1} \frac{1}{x^{2/3}+x^{1/3}+1} = \frac{1}{3}$

ALTERNATIVNO: $\lim\limits_{x\to 1} \frac{x^{1/3}-1}{x-1} \overset{\frac{0}{0}}{=} \lim \frac{\frac{1}{3}x^{-\frac{2}{3}}-0}{1-0} = \frac{1}{3}\cdot 1$
(L'HP)

$x^{a\,\prime} = a\,x^{a-1}$

(f) $\lim\limits_{x\to 0} \frac{\sin(5x)}{\sin(x)} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \frac{\sin(5x)\cdot 5x \cdot x}{\sin(x)\cdot 5x \cdot x}$

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ $\quad = \lim\limits_{x\to 0} \frac{\sin(5x)}{5x} \cdot \lim \frac{x}{\sin x} \cdot \lim \frac{5\cancel{x}}{\cancel{x}} = 1\cdot 1\cdot 5$

$5x = t \qquad \lim\limits_{x\to 0}\frac{\sin(5x)}{5x} = \lim\limits_{t\to 0} \frac{\sin t}{t} \qquad \lim \frac{x}{\sin x} = 1$

ALTERNATIVNO: $\lim\limits_{x\to 0} \frac{\cos(5x)\cdot 5}{\cos(x)} = \frac{1\cdot 5}{1} = 5$
(L'HP)

$\boxed{f(g(x))' = f'(g(x))\cdot g'(x)}$

(g) $\lim\limits_{x\to 0} \frac{x\,\mathrm{tg}(x)}{1-\cos x} \overset{\frac{0}{1-1}}{=} \lim\limits_{x\to 0} \frac{x\cdot\sin(x)}{\cos(x)(1-\cos x)} = \lim \frac{x\sin x}{1-\cos x}$

$\cos(x) \to 1$

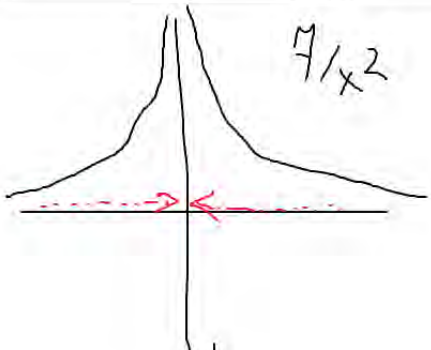
$\overset{L'HP}{=} \lim \frac{1\cdot\sin x + x\cos x}{0-(-\sin x)} = \lim\limits_{x\to 0} \frac{\sin x + x\cos x}{\sin x} \overset{\frac{0}{0}}{=}$

$\boxed{(f\cdot g)' = f'\cdot g + f g'}$

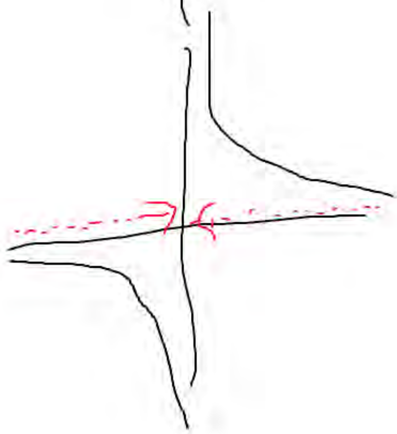
$\overset{L'HP}{=} \lim\limits_{x\to 0} \frac{\cos x + 1\cos x - x\sin x}{\cos x}$

$= \frac{1+1-0}{1} = 2$

$\lim\limits_{x\to 0} \frac{7}{x^2} \overset{\frac{7}{+0}}{=} +\infty$



$\lim\limits_{x\to 0} \frac{7}{x}$ // limita ne obstaja

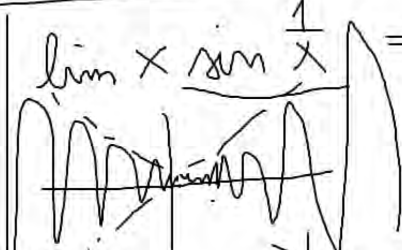


$\lim\limits_{x\uparrow 0} \frac{7}{x} = -\infty$

$\lim\limits_{x\downarrow 0} \frac{7}{x} = +\infty$

$\lim\limits_{x\to 0} \frac{1}{\sin\frac{1}{x}}$

$\lim x \sin\frac{1}{x} = 0$



$\lim\limits_{n\to\infty} (-1)^n \nexists$

$\lim \frac{(-1)^n}{n} = 0$

2.(g)

$$\lim_{x\to 0}\frac{x\,\mathrm{tg}\,x}{1-\cos x}=\lim_{x\to 0}\frac{x\sin x}{\cos x(1-\cos x)}=\lim_{x\to 0}\frac{1}{\cos x}\cdot\lim_{x\to 0}\frac{x\sin x}{1-\cos x}$$

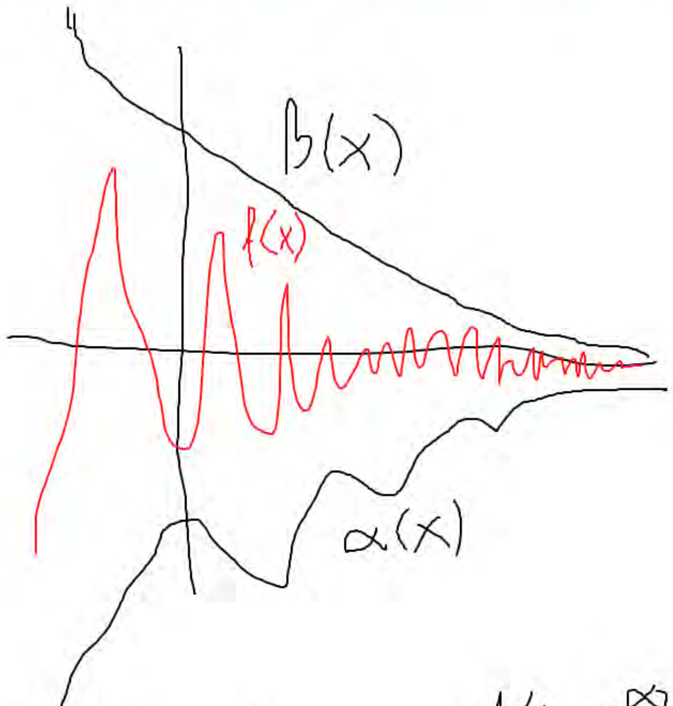
$$=1\cdot\lim\frac{x\sin x\cdot(1+\cos x)}{(1-\cos x)(1+\cos x)}=\lim\frac{x\cdot\cancel{\sin x}\,(1+\cos x)}{\sin^{\cancel{2}}x}=\frac{2}{1}$$

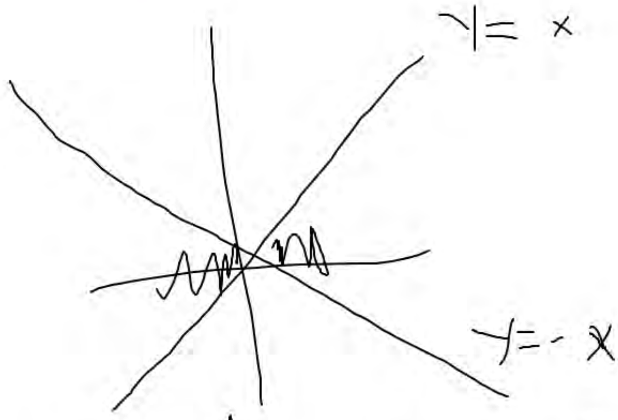
(h) $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)=0$, $\quad \frac{1}{x}\to\infty$

$$-1\le\sin(\cdot)\le 1\Rightarrow \underset{\to 0}{-x}\le x\sin\left(\tfrac{1}{x}\right)\le \underset{\to 0}{x}$$

IZREK O SENDVIČU: $\alpha(x)\le f(x)\le\beta(x)\Rightarrow \lim f(x)=C$

$$\alpha(x)\xrightarrow{x\to a}C\xleftarrow{x\to a}\beta(x)$$





(i) $\lim_{x\to 0}(1+x+x^2)^{1/x}\overset{1^\infty}{=}\lim_{t\to\pm\infty}\left(1+\frac{1}{t}+\frac{1}{t^2}\right)^t=\lim_{t\to\infty}\left(1+\frac{t+1}{t^2}\right)^t$

$t=\frac{1}{x}$

$x\to 0\Rightarrow t\to\pm\infty$

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e}$$

$$=\lim\left(1+\frac{1}{t^2/(t+1)}\right)^{\frac{t^2}{t+1}\cdot\frac{(t+1)\cdot t}{t^2}}$$

$$=\left(\lim\left(1+\frac{1}{t^2/(t+1)}\right)^{\frac{t^2}{t+1}}\right)^{\frac{(t+1)\cdot t}{t^2}}=$$

Če $\lim_{x\to a}f(x)=\pm\infty$,

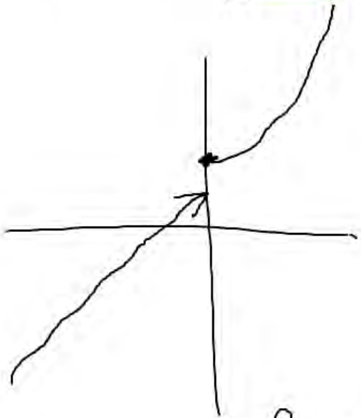
potem $\lim_{x\to a}\left(1+\frac{1}{f(x)}\right)^{f(x)}=e$

$f(t)=\frac{t^2}{t+1}\xrightarrow{t\to\pm\infty}\pm\infty$

$\lim_{t\to\pm\infty}\frac{t^2+t}{t^2}\overset{:t^2}{=}\lim\frac{1+\frac{1}{t}}{1}=1$

$=e^1$

4. $f(x)=\begin{cases} x^2+1; & x \geq 0 \\ x+a; & x<0 \end{cases}$ pri katerih a je f. ZVZ?



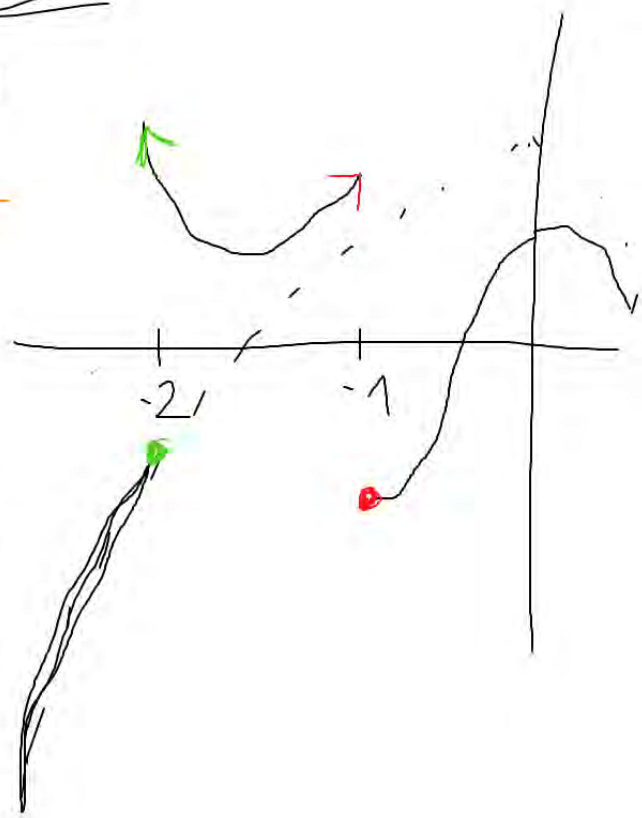
f je ZVZ v a $\Longleftrightarrow$

$$\lim_{x\uparrow a} f(x) = f(a) = \lim_{x\downarrow a} f(x)$$

$$\lim_{x\uparrow 0} f(x) = 0+a \overset{?}{=} f(0) = 1 = \lim_{x\downarrow 0} f(x) = 0^2+1$$

$$\underline{\underline{a = 1 = 1}}$$

5. $f(x)=\begin{cases} 1-x^2 & ;\ \text{če } x \leq -2 \\ x^2+ax+b & ;\ \text{če } -2<x<-1 \\ \cos(\pi x) & ;\ \text{če } -1 \leq x \end{cases}$



f je ZVZ $\Longleftrightarrow$

$\lim_{x\uparrow -2} f(x) = \lim_{x\downarrow -2} f(x)$ in $\lim_{x\uparrow -1} f(x) = \lim_{x\downarrow -1} f(x)$

$1-(-2)^2 = (-2)^2 - 2a + b$ in $(-1)^2 - a + b = -1$

$-3 = 4 - 2a + b$ $\qquad -a+b=-2$

$-a+b=-2$

$-2a+b=-7$

$(1)-(2):\ a = -2+7 = \underline{\underline{5}}$

$b = a-2 = \underline{\underline{3}}$

6 $f(x) = x^7 - x + 1$ ničlo na $[-2, 0]$?

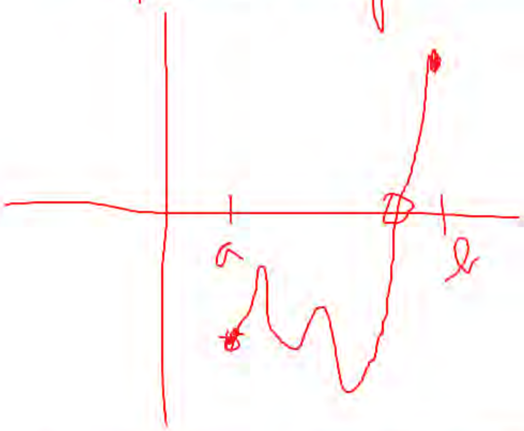
$f(-2) = (-2)^7 + 2 + 1 = -125$

$f(0) = 1$

Če je f na $[a,b]$ ZVZ

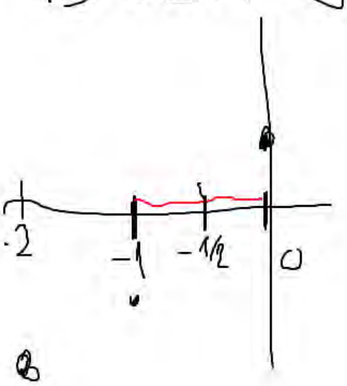
in $f(a) \cdot f(b) < 0$

$\Rightarrow$ f ima vsaj 1 ničlo na $[a,b]$



ker sta $f(-2)$ in $f(0)$ različnih predznakov, ter ker je f ZVZ (saj je elementarna), vemo, da ima f vsaj 1 ničlo na $[-2, 0]$

BISEKCIJA (numerično računanje ničel)



DODATNE NALOGE:

(a) $\lim_{x\to\infty}\frac{x-1}{x^3-1} \overset{:x^3}{\underset{:x^3}{=}} \lim\frac{\frac{1}{x^2}-\frac{1}{x^3}}{1-\frac{1}{x^3}} = \frac{0-0}{1-0} = \underline{\underline{0}}$

(b) $\lim_{x\to1}\frac{x-1}{x^3-1} \overset{\frac{0}{0}}{=} \lim_{x\to1}\frac{\cancel{x-1}}{\cancel{(x-1)}(x^2+x+1)} = \frac{1}{1+1+1} = \underline{\underline{\frac{1}{3}}}$

(c) $\lim_{x\to2}\frac{x^3-8}{x^2-4} = \lim_{x\to2}\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)} = \frac{4+4+4}{4} = \underline{\underline{3}}$

(d) $\lim_{x\to1}\frac{\sqrt{3+x}-2}{x-1} \overset{\frac{2-2}{0}}{=} \lim_{x\to1}\frac{(\sqrt{3+x}-2)}{(x-1)}\frac{(\sqrt{3+x}+2)}{(\sqrt{3+x}+2)} = \lim\frac{\cancel{3+x-2^2}}{\cancel{(x-1)}(\sqrt{3+x}+2)}$

$= \frac{1}{\sqrt{3+1}+2} = \underline{\underline{\frac{1}{4}}}$

(e) $\lim_{x\to\infty}\left(\underbrace{\frac{x^2+1}{x^2-1}}_{\to 1}\right)^{x} \overset{1^\infty}{=} \lim\left(1+\frac{1}{(x^2-1)/2}\right)^{x} =$

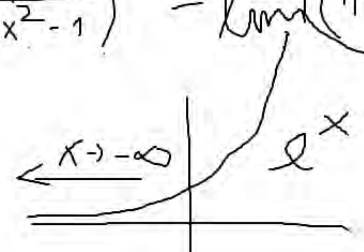
$\left(1+\frac{1}{f(x)}\right)^{f(x)} \xrightarrow{f(x)\to\pm\infty} e$

$$\left\{\begin{aligned} 1+\frac{1}{f} &= \frac{x^2+1}{x^2-1} \\ \frac{1}{f} &= \frac{(x^2+1)-(x^2-1)}{x^2-1} = \frac{2}{x^2-1} \\ f &= \frac{x^2-1}{2} \end{aligned}\right.$$

$\lim_{x\to\infty}\underbrace{\left(1+\frac{1}{(x^2-1)/2}\right)^{\frac{x^2-1}{2}}}_{\to e}{}^{\cdot\overbrace{\frac{2\cdot x}{x^2-1}}^{\to 0}} = e^{0} = \underline{\underline{1}}$

- $\lim\left(\frac{x^2+1}{x^2-1}\right)^{3x^2} \overset{1^\infty}{=} \lim\left(1+\frac{1}{(x^2-1)/2}\right)^{\frac{x^2-1}{2}\cdot\overbrace{\frac{2\cdot 3x^2}{x^2-1}}^{\to 6}} = \underline{\underline{e^{6}}}$
- $\lim\left(\frac{x^2+1}{x^2-1}\right)^{x^3/1000} \overset{1^\infty}{=} \lim\left(1+\frac{1}{(x^2-1)/2}\right)^{\frac{x^2-1}{2}\cdot\frac{2\cdot x^3}{(x^2-1)1000}} = e^{\infty} = \underline{\underline{\infty}}$
- $\lim\left(\frac{x^2-1}{x^2+1}\right)^{x^3} \overset{1^\infty}{=} \lim\left(\frac{x^2+1}{x^2-1}\right)^{-x^3} = \lim\underbrace{\left(1+\frac{1}{f}\right)^{f}}_{\to e}{}^{\cdot\underbrace{\frac{2\cdot(-x^3)}{x^2-1}}_{\to -\infty}}$

$= e^{-\infty} = \underline{\underline{0}}$



(l.) $\lim\limits_{x\to 0} \frac{tg(ax)}{tg(bx)} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0} \frac{\sin(ax)\cdot \cos(bx)}{\cos(ax)\cdot \sin(bx)} =$

$\cos(bx) \to 1$, $\cos(ax) \to 1$

$= \lim\limits_{x\to 0} \frac{\sin(ax)\cdot ax \cdot bx}{\sin(bx)\cdot ax \cdot bx} = \lim \frac{a\cancel{x}}{b\cancel{x}} = \frac{a}{b}$

$\to 1$, $\to 1$

$\overset{L'H}{=} \lim\limits_{x\to 0} \frac{\cos(ax)\cdot a}{\cos(bx)\cdot b} = \frac{a}{b}$

$\to 1$, $\to 1$

$\overset{L'H}{=} \lim \frac{1\cdot a \cdot \cos^2(bx)}{\cos^2(ax)\cdot 1 \cdot b} = \frac{a}{b}$

$\to 1$, $\to 1$