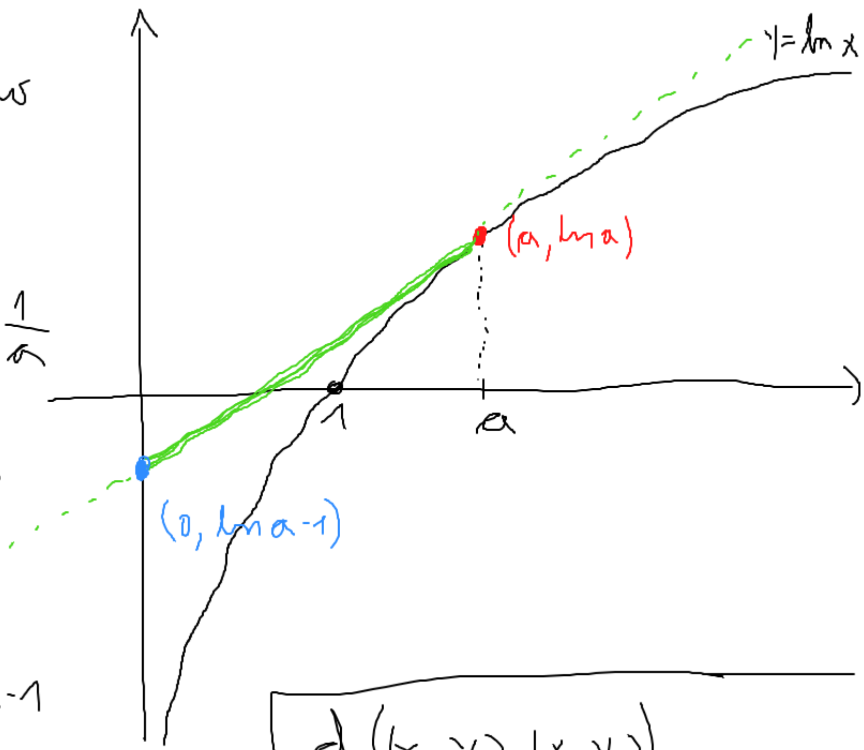
$\boxed{11}$ $\boxed{16.3.2021}$

$p \in (0,1)$, $x \in (0,\infty)$

v točki $(a, \ln a)$ izstrelimo kroglico s hitrostjo a^p



tangenta: $\frac{y-\ln a}{x-a} = f'(a) = \frac{1}{a}$

$$y = \frac{1}{a}(x-a) + \ln a$$

$$y = \frac{1}{a}x + \ln a - 1$$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$x = 0 \Rightarrow y = \ln a - 1$

dolžina razdalje: $\Delta s = \sqrt{a^2 + (\ln a - \ln a + 1)^2} = \sqrt{a^2+1}$

$$\boxed{d\big((x_0,y_0),(x_1,y_1)\big) = \sqrt{(x_0-x_1)^2 + (y_0-y_1)^2}}$$

$$a^p = \text{hitrost} = \overbrace{v = \frac{\Delta s}{t}}^{\text{PREMO ENAKOMERNO GIBANJE}} \Rightarrow \underset{\uparrow \atop \text{ČAS}}{t} = \frac{\Delta s}{v} = \frac{\sqrt{a^2+1}}{a^p} =: f(a)$$

Iščemo minimum od $f:(0,\infty) \to \mathbb{R}$

- $\lim\limits_{a\to\infty} \frac{\sqrt{a^2+1}}{a^p} \,\frac{:a}{:a} = \lim\limits_{a\to\infty} \frac{\sqrt{1+\frac{1}{a^2}}}{a^{p-1}} = \lim\limits_{a\to\infty} a^{1-p}\sqrt{1+\frac{1}{a^2}} = \infty \cdot \sqrt{1+0} = \infty$

  $p-1<0$ $\quad$ $1-p>0$

- $\lim\limits_{a\downarrow 0} \frac{\sqrt{a^2+1}}{a^p} = \frac{\sqrt{0^2+1}}{+0} = \frac{1}{+0} = +\infty$

- $f'(a) = \frac{\frac{1}{2}(a^2+1)^{-1/2}\cdot 2a \cdot a^p - \sqrt{\ }\; p a^{p-1}}{a^{2p}} = \frac{\frac{a^{p+1}}{\sqrt{\ }} - \sqrt{\ }\; p a^{p-1}}{a^{2p}} = \frac{a^{p+1} - (a^2+1)p a^{p-1}}{a^{2p}\sqrt{a^2+1}}$

$f' = 0 \iff a^{p+1} = (a^2+1)p a^{p-1} \;/ : a^{p-1} \iff a^2 = (a^2+1)p \iff a^2(1-p) = p$

$\iff a^2 = \frac{p}{1-p} \iff a = +\underline{\underline{\sqrt{\frac{p}{1-p}}}}$

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PREMO ENAKOMERNO GIBANJE:

$v = \frac{s}{t}$

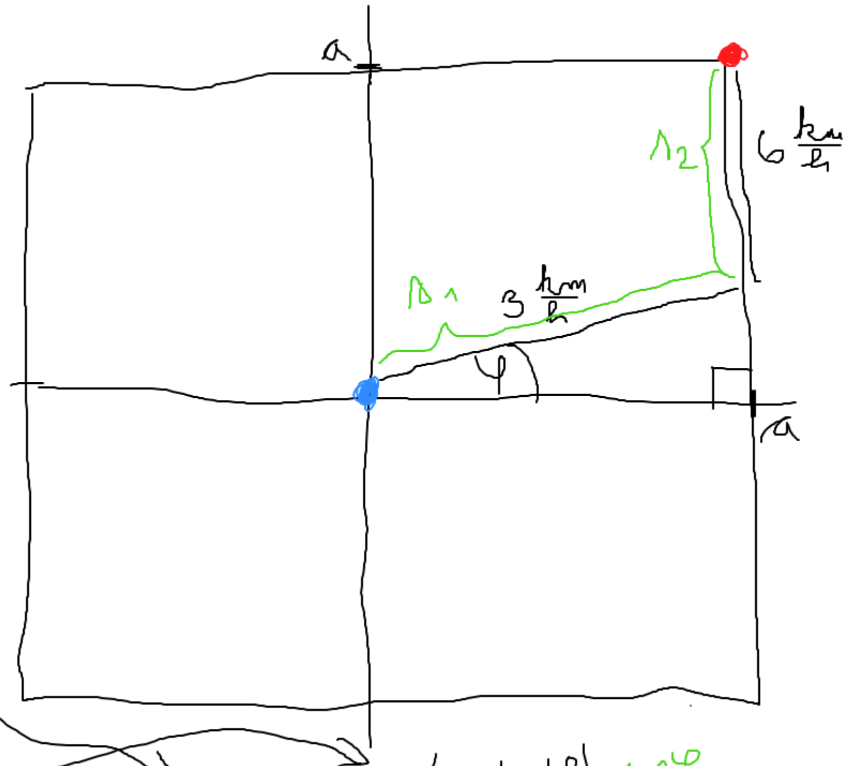
$s_1 = ?$ $\quad s_2 = ?$

$\cos\varphi = \frac{a}{s_1}$, $\quad \operatorname{tg}\varphi = \frac{a - s_2}{a}$

$\Rightarrow s_1 = \frac{a}{\cos\varphi}$, $\quad s_2 = a - a\operatorname{tg}\varphi$

$\Rightarrow t_1 = \frac{s_1}{v_1} = \frac{a}{v_1 \cos\varphi}$

$t_2 = \frac{s_2}{v_2} = \frac{a(1-\operatorname{tg}\varphi)}{v_2}$



minimiziramo čas $t = t_1 + t_2 = \frac{a \cdot 2}{3\cos\varphi \cdot 2} + \frac{a(1-\operatorname{tg}\varphi)\cdot\cos\varphi}{6 \cdot \cos\varphi} =: f(\varphi)$

pri katerem φ je $f(\varphi)$ minimalen?

$$f(\varphi) = \frac{a(2+\cos\varphi-\sin\varphi)}{6\cos\varphi}, \quad \varphi \in \left[0, \frac{\pi}{4}\right]$$

- pogledamo vrednost v robnih točkah:

$f(0) = \frac{a \cdot 3}{6} = \frac{a}{2}$, $\quad f\left(\frac{\pi}{4}\right) = \frac{a\left(2 + \cancel{\frac{1}{\sqrt2}} - \cancel{\frac{1}{\sqrt2}}\right)}{6 \cdot 1/\sqrt2} = \frac{a\sqrt2}{3} \Rightarrow f\left(\frac{\pi}{4}\right) < f(0)$

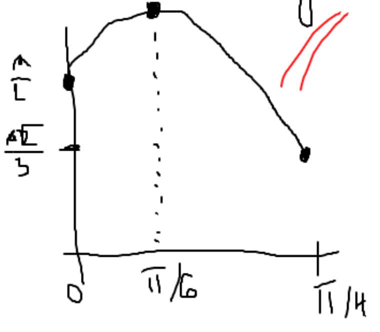
- $f' = \frac{a}{6}\,\frac{(-\sin\varphi - \cos\varphi)\cdot\cos\varphi - (2+\cos\varphi-\sin\varphi)\cdot(-\sin\varphi)}{\cos^2\varphi} = \frac{a}{6}\,\frac{-\cos^2\varphi + 2\sin\varphi - \sin^2\varphi}{\cos^2\varphi}$

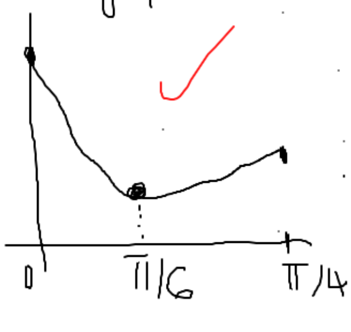
$= \frac{a}{6}\,\frac{2\sin\varphi - 1}{\cos^2\varphi}$

kandidati za ekstreme (minimume) so stacionarne točke: $f' = 0 \Leftrightarrow 2\sin\varphi - 1 = 0$

$\Leftrightarrow \sin\varphi = \frac{1}{2} \Leftrightarrow \boxed{\varphi = \frac{\pi}{6}}$

kateri od grafov za f je pravilen?





$f\left(\frac{\pi}{6}\right) = \frac{a\left(2+\frac{\sqrt3}{2}-\frac{1}{2}\right)}{6\,\frac{\sqrt3}{2}} = \frac{a(4+\sqrt3-1)}{6\sqrt3}$

$= \frac{a(3+\sqrt3)}{6\sqrt3} \begin{matrix} \cdot\sqrt3 \\ \cdot\sqrt3 \end{matrix} = \frac{a(\sqrt3+1)}{6}$

$\frac{a(\sqrt3+1)}{6} \overset{?}{<} \frac{a\sqrt2}{3}$

$\sqrt3 + 1 < 3\sqrt2$ ✓, torej je pri $\varphi = \frac{\pi}{6}$ minimum

19. 3. 2021

MEAN VALUE THEOREM

IZREK (LAGRANGE): $f:[a,b]\to\mathbb{R}$ zvezna, f odvedljiva na (a,b)

$$\Rightarrow \boxed{\frac{f(b)-f(a)}{b-a} = f'(t)} \text{ za neki } t\in(a,b)$$

13 RBD: $\forall x,y$ velja $|\operatorname{atg} x - \operatorname{atg} y| \le |x-y|$

$$\Longleftrightarrow \frac{|\operatorname{atg} x - \operatorname{atg} y|}{|x-y|} \le 1$$

$$\Longleftrightarrow \left|\frac{\operatorname{atg} x - \operatorname{atg} y}{x-y}\right| \le 1$$

$$f(x) := \operatorname{atg} x \overset{MVT}{\Longrightarrow} \left|\frac{\operatorname{atg} x - \operatorname{atg} y}{x-y}\right| = |f'(t)| \text{ za neki } t\in[x,y]$$

$$f'(x) = \frac{1}{1+x^2} \qquad = \left|\frac{1}{1+t^2}\right| = \frac{1}{1+t^2} \le 1 \quad \checkmark$$

15 L'HOSPITAL: če $f(x), g(x) \xrightarrow{x\to a} 0$ ali $f(x), g(x) \xrightarrow{x\to a} \pm\infty$ in če $g'(x)\neq 0$ na $(a-\varepsilon,a)\cup(a,a+\varepsilon)$, če f in g odvedljivi, če $\exists \lim_{x\to a}\frac{f'}{g'}$,

potem $$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

(a) $$\lim_{x\to 0}\frac{\sin x}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{\cos x}{1} = \frac{\cos 0}{1} = \underline{\underline{1}}$$

(b) $$\lim_{x\to\infty}\frac{\overset{\text{POLINOM}}{p(x)}}{e^x} \overset{\frac{\pm\infty}{\infty}}{=} \lim_{x\to\infty}\frac{p'}{e^x} \overset{\frac{\pm\infty}{\infty}}{=} \lim \frac{p''}{e^x} = \dots = \lim\frac{p^{(n)}}{e^x} = \frac{C}{\infty} = 0$$

$p(x)$ stopnje n

$$\left(8x^{12} + \dots\right)^{(12)} = 8\cdot 12!$$

(c) $$\lim_{x\to 1}(1-x)\operatorname{tg}\left(\frac{\pi}{2}x\right)$$

$$0\cdot\pm\infty$$

$$= \lim_{x\to 1}\frac{(1-x)\overbrace{\sin\left(\frac{\pi}{2}x\right)}^{\to 1}}{\cos\left(\frac{\pi}{2}x\right)}$$

$$= 1\cdot\lim_{x\to 1}\frac{1-x}{\cos\left(\frac{\pi}{2}x\right)} \overset{\frac{0}{0}}{=} \lim_{x\to 1}\frac{-1}{-\sin\left(\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}} = \underline{\underline{\frac{2}{\pi}}}$$

(d) $\lim\limits_{x\to 0} (1-\cos x)\operatorname{ctg} x = \lim\limits_{x\to 0} \dfrac{(1-\cos x)\overbrace{\cos x}^{\to 1}}{\sin x} = 1\cdot \lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin x}$

$\quad\quad 0 \cdot (\pm\infty)$

$\stackrel{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x} = \dfrac{0}{1} = \underline{\underline{0}}$

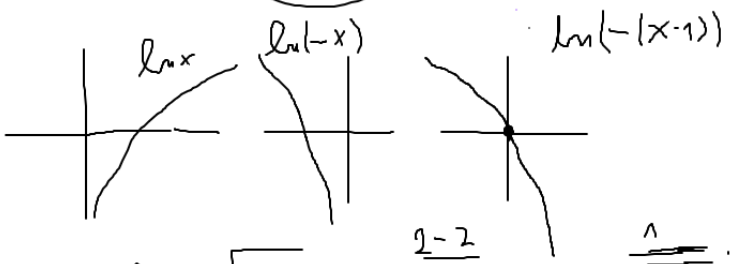
(e) $\lim\limits_{x\downarrow 0} x^2 \ln x = \lim\limits_{x\downarrow 0} \dfrac{\ln x}{x^{-2}} \stackrel{\frac{-\infty}{+\infty}}{=} \lim\limits_{x\downarrow 0} \dfrac{x^{-1}}{-2x^{-3}} = \lim\limits_{x\downarrow 0} \dfrac{x^2}{-2} = \underline{\underline{0}}$

$\quad\quad 0\cdot(-\infty)$

DODATNE NALOGE

$\boxed{1}$ (a) $\lim\limits_{x\to 0} \dfrac{e^x-1}{\sin 2x} \stackrel{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{e^x}{2\cos 2x} = \dfrac{1}{2\cdot 1} = \underline{\underline{\dfrac{1}{2}}}$

(b) $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\ln(1-x)} \stackrel{\frac{0}{0}}{=} \lim\limits_{x\to 0} \dfrac{3\cos(3x)}{\frac{1}{1-x}\cdot(-1)} = \dfrac{3}{1\cdot(-1)} = \underline{\underline{-3}}$



(c) $\lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2}{x-1} \stackrel{\frac{2-2}{1-1}}{=} \lim\limits_{x\to 1} \dfrac{\frac{1}{2\sqrt{x+3}}\cdot 1}{1} = \underline{\underline{\dfrac{1}{4}}}$

$\sqrt{x}\,' = {x^{1/2}}' = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

(d) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{x+1}-2}{\sqrt{x+4}-3} = \dfrac{1-2}{2-3} = \underline{\underline{1}}$

(e) $\lim\limits_{x\to 2} \dfrac{\sqrt[3]{x^2+4}-2}{\sqrt{4x+1}-3} \stackrel{\frac{0}{0}}{=} \lim \dfrac{(x^2+4)^{1/3}-2}{(4x+1)^{1/2}-3} = \lim \dfrac{\frac{1}{3}(x^2+4)^{-2/3}\cdot \cancel{2}x}{\cancel{\frac{1}{2}}(4x+1)^{-1/2}\cdot \cancel{4}}$

$= \lim\limits_{x\to 2} \dfrac{(4x+1)^{1/2}\cdot x}{3(x^2+4)^{2/3}} = \dfrac{3\cdot 2}{3\cdot 4} = \underline{\underline{\dfrac{1}{2}}}$

(f) $\lim\limits_{x\to\infty} \dfrac{x-\sin x}{2x+\sin x} \begin{matrix} :x \\ :x \end{matrix} \stackrel{\frac{\infty}{\infty}}{=} \lim\limits_{x\to\infty} \dfrac{1-\overbrace{\frac{\sin x}{x}}^{\to 0}}{2+\underbrace{\frac{\sin x}{x}}_{\to 0}} = \dfrac{1}{2}$

$-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$

$x\to\infty$

0

g) $\lim\limits_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x} \overset{\frac{1-1-0}{0-0}}{=} \lim\limits_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x} \overset{\frac{1+1-2}{1-1}}{=} \lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\sin x} \overset{\frac{1-1}{0}}{=}$

$= \lim\limits_{x\to 0}\frac{e^x+e^{-x}}{\cos x} = \frac{1+1}{1} = \underline{\underline{2}}$

PRIMERI, KJER L'HOPITAL ZAKOMPLICIRA:

- $\lim\limits_{x\to 0}\frac{e^{x^2}-1}{1-\cos x} \overset{\frac{1-1}{1-1}}{=} \lim\limits_{x\to 0}\frac{e^{x^2}\cdot 2x}{\sin x} \overset{\frac{0}{0}}{=} e^0\cdot 2 = \underline{\underline{2}}$

  $\frac{2x}{\sin x} \to 1$

- $\lim\limits_{x\to 0}\frac{\sqrt{1+x^2}\cdot\cos x-1}{x^4} \overset{\frac{1-1}{0}}{=} \lim \frac{\frac{2x}{2\sqrt{\ }}\cos x-\sqrt{\ }\sin x}{4x^3} = \lim \frac{\frac{x\cos x-\sqrt{\ }^2\sin x}{\sqrt{\ }}}{4x^3}$

  $= \lim\limits_{x\to 0}\frac{x\cos x-(1+x^2)\sin x}{4\sqrt{1+x^2}\,x^3} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\frac{\cos x-x\sin x-2x\sin x-(1+x^2)\cos x}{4\frac{x}{\sqrt{\ }}\cdot x^3+4\sqrt{\ }\cdot 3x^2}$

  $\frac{1-0-0-1}{0} =$

- $\lim\limits_{x\to 0}\frac{\operatorname{tg} x-x}{x-\sin x} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\frac{\frac{1}{\cos^2 x}-1}{1-\cos x} \overset{\frac{1-1}{1-1}}{=} \lim\frac{\cos^{-2}x-1}{1-\cos x} = \lim\frac{+2\cos^{-3}\cdot\cancel{\sin x}}{\cancel{\sin x}}$

  $= \lim\limits_{x\to 0}\frac{2}{\cos^3 x} = 2$