INTEGRALI

16.4.2021

$$\int f = F \iff f = F'$$

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g \quad (\alpha, \beta \in \mathbb{R})$$

$$\boxed{\text{PER PARTES: } \int f \cdot g' = f \cdot g - \int f' \cdot g}$$

$$\int x^a = \begin{cases} \frac{x^{a+1}}{a+1} & ; a \neq -1 \\ \ln|x| & ; a = -1 \end{cases} + C$$

$$\int \sin x = -\cos x + C$$

$$\int \cos x = \sin x + C$$

$$\int e^x = e^x + C$$

$\boxed{1}$ (a) $\int \left(x^2 - x^3 + \frac{x^7}{4} - \sqrt{x^3} + \frac{1}{\sqrt[3]{x^2}}\right) dx$

$$= \int \left(x^2 - x^3 + \frac{1}{4}x^7 - x^{3/2} + x^{-2/3}\right) dx$$

$$= \frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{4} \cdot \frac{x^8}{8} - \frac{x^{5/2}}{5/2} + \frac{x^{1/3}}{1/3} + C$$

(b) $\int \left(2 \sin x - \frac{1}{\cos^2 x} + \frac{2}{\sqrt{1-x^2}} + \frac{4}{1+x^2}\right) dx = -2\cos x - \operatorname{tg} x + 2 \arcsin x + 4 \operatorname{arctg} x + C$

(c) $\int e^{x+1} dx = e^{x+1} + C \leftarrow$ ker $(e^{x+1} + C)' = e^{x+1} \cdot 1$ ✓

$$\int e^{2x} dx = \frac{e^{2x}}{2} + C \leftarrow \text{ker } \left(\frac{e^{2x}}{2} + C\right)' = \frac{1}{\cancel{2}} e^{2x} \cdot \cancel{2} \checkmark$$

(d) $\int a^{2x} dx = \underbrace{\int (a^2)^x dx}_{a^2 = b} = \int b^x dx = \int e^{\ln(b^x)} dx = \underbrace{\int e^{x \cdot \ln b} dx}_{c := \ln b} =$

$$= \int e^{cx} dx = \frac{e^{cx}}{c} + C = \frac{e^{x \ln b}}{\ln b} + C = \frac{b^x}{\ln b} + C = \frac{(a^2)^x}{\ln(a^2)} + C$$

$$= \frac{a^{2x}}{2 \ln a} + C$$

(e) $\int \frac{\cos 2x}{\cos^2 x} dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x} dx = \int \frac{\cos^2 x - (1 - \cos^2 x)}{\cos^2 x}$

$$= \int \frac{2\cos^2 x - 1}{\cos^2 x} dx = \int \left(2 - \frac{1}{\cos^2 x}\right) dx = 2x - \operatorname{tg} x + C$$

$\boxed{2}$ (a) $\int (x-1)^{500} dx = \int t^{500} \, dt = \frac{t^{501}}{501} + C = \frac{(x-1)^{501}}{501} + C$

$$t = x - 1$$
$$1 \cdot dt = 1 \cdot dx$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

ALTERNATIVNO:

$$\int \sum_{k=0}^{500} \binom{500}{k} x^k \underbrace{(-1)^{500-k}}_{(-1)^k} = \sum_{k=0}^{500} \binom{500}{k} (-1)^k \cdot \int x^k = \sum_{k=0}^{500} \binom{500}{k} (-1)^k \cdot \frac{x^{k+1}}{k+1} + C$$

(b) $\int \sin(2x)\,dx = \int \sin t \cdot \frac{dt}{2} = \frac{1}{2}\int \sin t\,dt = \frac{1}{2}(-\cos t)+C = \frac{-\cos 2x}{2}+C$

$2x = t$

$2\cdot dx = 1\cdot dt$

(c) $\int \operatorname{tg} x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{t}{\cos^2 x}\,dt = \int \frac{t\,dt}{1-t^2} = \int \frac{\frac{ds}{-2}}{s} = \frac{-1}{2}\int s^{-1}\,ds$

$t = \sin x$

$1\cdot dt = \cos x \cdot dx$

$dx = \frac{dt}{\cos x}$

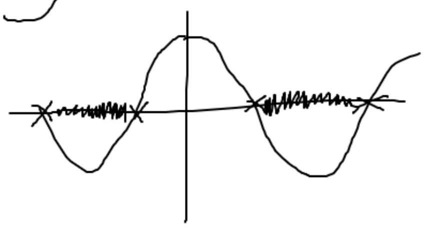
$1-t^2 = s$

$-2t\,dt = ds$

$= -\frac{1}{2}\ln s + C$

$= -\frac{1}{2}\ln(1-t^2)+C$

$= -\frac{1}{2}\ln(\cos^2 x)+C$



$\neq -\ln \cos x + C$

BOLJŠI NAČIN:

$\int \frac{\sin x}{\cos x}\,dx = \int \frac{-dt}{t} = -\ln t + C = -\ln \cos x + C$

$t = \cos x$

$dt = -\sin x\,dx$

(d) $\int \frac{dx}{x\ln^2 x} = \int \frac{dt}{t^2} = \int t^{-2}\,dt = \frac{t^{-1}}{-1}+C = \frac{-1}{t}+C = \frac{-1}{\ln x}+C$

$t = \ln x$

$dt = \frac{1}{x}\,dx$

(e) $\int x e^{x^2}\,dx = \int e^t \frac{dt}{2} = \frac{1}{2}\int e^t\,dt = \frac{1}{2}e^t + C = \frac{e^{x^2}}{2}+C$

$t = x^2$

$dt = 2x\,dx$

(f) $\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 t}\,\cos t\,dt = \int \sqrt{\cos^2 t}\,\cos t\,dt \overset{\text{če } \cos t \geq 0}{=} \int \cos^2 t\,dt =$

$x = \sin t \longrightarrow dx = \cos t\,dt$

$t = \arcsin x$

$dt = \frac{1}{\sqrt{1-x^2}}\,dx$

$\cos 2x = \cos^2 x - \sin^2 x$

$= (1-\sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$

$= \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$

$= \frac{1}{2}\int (1+\cos 2t)\,dt = \frac{1}{2}\left(t + \int \cos(2t)\,dt\right)$

$s = 2t,\ ds = 2\,dt$

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}}$$

$= \frac{1}{2}\left(t + \int \cos s \cdot \frac{ds}{2}\right) = \frac{1}{2}\left(t + \frac{1}{2}\sin s\right) + C = \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) + C$

$= \frac{t}{2} + \frac{\sin 2t}{4} + C = \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + C =: F$

$\cos(\cdot) = \pm\sqrt{1-\sin^2(\cdot)}$

PREIZKUS: $F' = \frac{1}{2}\frac{1}{\sqrt{1-x^2}} + \frac{1}{4}\cdot\cos(2\arcsin x)\cdot\frac{2}{\sqrt{1-x^2}} = \frac{1}{2}\cdot\frac{1+\cos(2\arcsin x)}{\sqrt{1-x^2}} =$

$= \frac{1 \pm \sqrt{1-\sin^2(2\arcsin x)}}{2\sqrt{1-x^2}} =$

$$F = \frac{t}{2} + \frac{\sin 2t}{4} + C = \frac{t}{2} + \frac{2\,\overset{=x}{\sin t}\,\overset{=\pm\sqrt{1-\sin^2 t}}{\cos t}}{4} + C =$$

$$= \frac{\operatorname{asin} x + x \cdot \sqrt{1-\sin^2 \operatorname{asin} x}}{2} = \frac{1}{2}\left(\operatorname{asin} x + x\sqrt{1-x^2}\right) + C$$

$$F' = \frac{1}{2}\left(\frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} + x\,\frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\right) = \frac{1+(1-x^2)-x^2}{2\sqrt{1-x^2}} = \frac{2-2x^2}{2\sqrt{1-x^2}} =$$

$$= \frac{1-x^2}{\sqrt{1-x^2}} = \frac{\sqrt{1-x^2}^{\,2}}{\sqrt{1-x^2}} = \underline{\underline{\sqrt{1-x^2}}}$$