$\boxed{3}$ **20.4.2021**

PER PARTES: $\boxed{\int f\cdot g' = f\cdot g - \int f' g}$

(a) $$\int \overset{g'}{x}\,\overset{f}{\ln x}\,dx = \frac{x^2}{2}\cdot \ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + C = \frac{x^2}{4}(2\ln x - 1) + C$$

(b) $$\int \overset{g'}{1}\cdot\overset{f}{\ln x}\,dx = x\ln x - \int \overbrace{x\cdot\frac{1}{x}}^{1} = x\ln x - x + C = x(\ln x - 1) + C$$

(c) $$\int \overset{g'}{x}\,\overset{f}{\cos(2x)}\,dx = \frac{x^2}{2}\cdot\cos(2x) - \int \frac{x^2}{2}\cdot(-\sin(2x)\cdot\cancel{2})\,dx$$
$$\int \overset{f}{x}\,\overset{g'}{\cos(2x)}\,dx = x\cdot\frac{\sin 2x}{2} - \int 1\cdot\frac{\sin 2x}{2}\,dx$$
$$= x\frac{\sin 2x}{2} + \frac{1}{2}\int -\sin 2x\,dx = x\frac{\sin 2x}{2} + \frac{1}{2}\,\frac{\cos 2x}{2} + C$$

(d) $$\int \overset{f}{x^2}\,\overset{g'}{e^x}\,dx = x^2e^x - \int \underset{f}{2x}\cdot\underset{g'}{e^x}\,dx = x^2e^x - 2\left(xe^x - \int 1\cdot e^x\,dx\right)$$
$$= x^2e^x - 2xe^x + 2e^x + C = (x^2-2x+2)e^x + C$$

(e) $$\int \overbrace{(x^3-3x+1)}^{f}\,\overbrace{e^x}^{g'}\,dx = \int p\cdot e^x = p\cdot e^x - \int p'e^x = pe^x - \left(p'e^x - \int p''e^x\right)$$
$$= pe^x - p'e^x + \int p''e^x = pe^x - p'e^x + p''e^x - \int p'''e^x =$$
$$= e^x(p - p' + p'' - p''') + \int p''''e^x = \dots = \boxed{e^x(\underbrace{p - p' + p'' - p''' + p'''' - \dots}_{})}$$

p polinom $\Rightarrow$ končna vrsta

v našem primeru:

$$= e^x\left((x^3-3x+1) - (3x^2-3) + (6x) - 6\right) + C = e^x(x^3-3x^2+3x-2) + C$$

(f) $$I = \int \underset{f}{e^{2x}}\,\underset{g'}{\sin x}\,dx = e^{2x}(-\cos x) - \int e^{2x}\cdot 2\cdot(-\cos x)\,dx$$
$$= -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$$
$$= -e^{2x}\cos x + 2\left(e^{2x}\sin x - \int e^{2x}\cdot 2\cdot\sin x\,dx\right)$$
$$= e^{2x}(2\sin x - \cos x) - 4\underbrace{\int e^{2x}\sin x\,dx}_{I}$$

$$\Rightarrow I = e^{2x}(2\sin x - \cos x) - 4I$$
$$5I = \dots$$
$$I = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

(g) $\int 1\cdot\sqrt{1+x^2}\,dx \overset{P.P.}{=} x\cdot\sqrt{1+x^2} - \int x\cdot\frac{2x}{2\sqrt{1+x^2}}\,dx$

$x = \operatorname{tg} t$

$dx = \frac{dt}{\cos^2 t}$

$\int \sqrt{1+\operatorname{tg}^2 t}\,\frac{dt}{\cos^2 t} = \int \frac{1}{\cos t}\cdot\frac{dt}{\cos^2 t} = \int \frac{\cos t\,dt}{\cos^4 t} = \int \frac{dr}{(1-r^2)^2} = \dots$

$1+\frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$

$r = \sin t$

$dr = \cos t\,dt$

$\operatorname{tg} t\cdot\frac{1}{\cos t} - \int \operatorname{tg} t\cdot\frac{-1}{\cos^2 t}\cdot(-\sin t)\,dt$

$= \operatorname{tg} t\cdot\frac{1}{\cos t} - \underbrace{\int \frac{\sin^2 t}{\cos^3 t}\,dt}$

$= \int \frac{\sin^2 t\,\cos t\,dt}{\cos^4 t} = \int \frac{r^2\,dr}{(1-r^2)^2} = \dots$

$r = \sin t$

$dr = \cos t\,dt$

$\operatorname{sh} t = \frac{e^t - e^{-t}}{2}$, velja $\operatorname{sh}(-t) = \frac{e^{-t}-e^{t}}{2} = -\frac{e^t - e^{-t}}{2} = -\operatorname{sh} t$ ... liha

$\operatorname{ch} t = \frac{e^t + e^{-t}}{2}$, $\operatorname{ch}(-t) = \frac{e^{-t}+e^{t}}{2} = \frac{e^t + e^{-t}}{2} = \operatorname{ch} t$ ... soda

$\operatorname{ch}^2 t - \operatorname{sh}^2 t = 1$

$\left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{\cancel{e^{2t}}+2+\cancel{e^{-2t}}}{4} - \frac{\cancel{e^{2t}}-2+\cancel{e^{-2t}}}{4} = \frac{2-(-2)}{4} = 1$ ✓

$\operatorname{sh} t' = \frac{1}{2}\left(e^t - e^{-t}\cdot(-1)\right) = \operatorname{ch} t$

$\operatorname{ch} t' = \frac{1}{2}\left(e^t + e^{-t}\cdot(-1)\right) = \operatorname{sh} t$

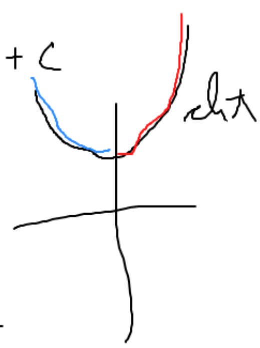
$x = \operatorname{sh} t$

$dx = \operatorname{ch} t\,dt$

$= \int \underbrace{\sqrt{1+\operatorname{sh}^2 t}}_{= \operatorname{ch}^2 t}\cdot\operatorname{ch} t\,dt = \int \sqrt{\operatorname{ch}^2 t}\,\operatorname{ch} t\,dt = \int \operatorname{ch}^2 t\,dt =$

$= \int \frac{e^{2t}+2+e^{-2t}}{4}\,dt = \frac{1}{4}\left(\frac{e^{2t}}{2} + 2t - \frac{e^{-2t}}{2}\right) + C = \frac{1}{4}\left(2t + \operatorname{sh} 2t\right) + C$

$= \frac{1}{4}\left(2\overset{=?}{\operatorname{arsh} x} + \operatorname{sh}(2\operatorname{arsh} x)\right) + C$



$x = \operatorname{sh} t = \frac{e^t - e^{-t}}{2}$

$2x = e^t - e^{-t} \quad /\cdot e^t$

$2x e^t = e^{2t} - 1 \quad / r = e^t$

$0 = r^2 - 2xr - 1 \quad a=1, b=-2x, c=-1$

$r = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1} = e^t$

$\Rightarrow t = \ln\left(x + \sqrt{x^2+1}\right) = \operatorname{arsh} x$

$x = \operatorname{ch} t = \frac{e^t + e^{-t}}{2}$

$2x = e^t + e^{-t} \quad /\cdot e^t$

$2x e^t = e^{2t} + 1 \quad / r = e^t$

$0 = r^2 - 2xr + 1$

$r = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1} = e^t$

$\Rightarrow t = \ln\left(x \pm \sqrt{x^2-1}\right) = \operatorname{arch} x$

23.4.2021

4 (a) $\int \frac{dx}{x-2} = \ln|x-2| + C$

(b) $\int \frac{dx}{ax+b} = \frac{\ln|ax+b|}{a} + C$ vr. $t = ax+b$, $dt = a\,dx$

(c) $\int \frac{dx}{x(x+1)} = \int \frac{1}{x} - \int \frac{1}{x+1} = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C$

$$\frac{1+0x}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{A + x(A+B)}{\text{-||-}}$$

$1:\ 1 = A \implies A = 1$

$x:\ 0 = A+B \quad B = -A = -1$

(d) $\int \frac{x-2}{x^3-x}\,dx = \int \frac{x-2}{x(x^2-1)}\,dx = \int \frac{(x-2)\,dx}{x(x-1)(x+1)} = 2\int\frac{1}{x} - \frac{1}{2}\int\frac{1}{x-1} - \frac{3}{2}\int\frac{1}{x+1} =$

$$\frac{x-2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x(x+1)(x-1)} = \frac{1(-A)+x(B-C)+x^2(A+B+C)}{\text{-||-}}$$

$1:\ -2 = -A \implies A = 2$

$x:\ 1 = B - C \implies B = C+1 = -1/2$

$x^2:\ 0 = A+B+C \implies 0 = 2 + C + 1 + C \implies C = -3/2$

$= 2\ln|x| - \frac{1}{2}\ln|x-1| - \frac{3}{2}\ln|x+1| + C = \ln\left|\frac{x^2}{\sqrt{(x-1)(x+1)^3}}\right| + C$

(e) $\int \frac{x-1}{x(x+1)^2}\,dx = -\int\frac{1}{x} + \int\frac{x+3}{(x+1)^2} = -\ln|x| + \int\frac{t+2}{t^2}\,dt = -\ln|x| + \int t^{-1} + 2\int t^{-2} =$

$t = x+1,\ dt = dx$

$$\frac{x-1}{x(x+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x+1)^2} = \frac{A(x^2+2x+1)+(Bx^2+Cx)}{\text{-||-}} = \frac{1(A)+x(2A+C)+x^2(A+B)}{\text{-||-}}$$

$1:\ -1 = A \implies A = -1$

$x:\ 1 = 2A + C \implies C = 1 - 2A = 1 + 2 = 3$

$x^2:\ 0 = A + B \implies B = -A = 1$

$= -\ln|x| + \ln|t| - 2t^{-1} + C$

$= -\ln|x| + \ln|x+1| - \frac{2}{x+1} + C$

(f) $\int \frac{5x^3-3x^2+2x-1}{\underbrace{x^4-5x^2+4}_{=(x^2-1)(x^2-4)=(x-1)(x+1)(x-2)(x+2)}}\,dx$

$$\frac{5x^3-3x^2+2x-1}{x^4-5x^2+4} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2} = \frac{A\overbrace{(x+1)(x^2-4)}^{x^3+x^2-4x-4} + B\overbrace{(x-1)(x^2-4)}^{x^3-x^2-4x+4} + C\overbrace{(x^2-1)(x+2)}^{x^3+2x^2-x-2} + D\overbrace{(x^2-1)(x-2)}^{x^3-2x^2-x+2}}{\text{-||-}}$$

$$= \frac{x^3(A+B+C+D) + x^2(A-B+2C-2D) + x(-4A-4B-C-D) + 1(-4A+4B-2C+2D)}{\text{-||-}}$$

$$\left.\begin{aligned}
1:&\ -1 = -4A+4B-2C+2D\\
x:&\ \ \ 2 = -4A-4B-C-D\\
x^2:&\ -3 = A-B+2C-2D\\
x^3:&\ \ \ 5 = A+B+C+D
\end{aligned}\right\}$$ 4×4 linearni sistem enačb

$$\begin{array}{c}
\begin{array}{cccc} A & B & C & D \end{array}\\
\left[\begin{array}{cccc|c}
-4 & 4 & -2 & 2 & -1\\
-4 & -4 & -1 & -1 & 2\\
1 & -1 & 2 & -2 & -3\\
1 & 1 & 1 & 1 & 5
\end{array}\right]
\end{array}
\sim
\begin{array}{l}(4)\\(1)+4(4)\\(2)+4(4)\\(3)-(4)\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5\\
0 & 8 & 2 & 6 & 19\\
0 & 0 & 3 & 3 & 22\\
0 & -2 & 1 & -3 & -8
\end{array}\right]
\sim
\begin{array}{l}(1)\\(4)\\(2)+4(4)\\(3)\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5\\
0 & -2 & 1 & -3 & -8\\
0 & 0 & 6 & -6 & -13\\
0 & 0 & 3 & 3 & 22
\end{array}\right]$$

$$\sim
\begin{array}{l}(1)\\(2)/2\\(4)\\(3)-2(4)\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5\\
0 & -1 & \frac{1}{2} & \frac{-3}{2} & -4\\
0 & 0 & 3 & 3 & 22\\
0 & 0 & 0 & -12 & -57
\end{array}\right]
\sim
\begin{array}{l}(1)\\(2)\\(3)/3\\(4)/12\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 5\\
0 & -1 & \frac{1}{2} & \frac{-3}{2} & -4\\
0 & 0 & 1 & 1 & \frac{22}{3}\\
0 & 0 & 0 & -1 & \frac{19}{4}
\end{array}\right]
\sim
\begin{array}{l}(1)+(4)\\(2)-\frac{3}{2}(4)\\(3)+(4)\\-(4)\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 0 & \frac{39}{4}\\
0 & -1 & \frac{1}{2} & 0 & -\frac{89}{8}\\
0 & 0 & 1 & 0 & \frac{145}{12}\\
0 & 0 & 0 & 1 & -\frac{19}{4}
\end{array}\right]$$

$$\sim
\begin{array}{l}(1)-(3)\\(2)-(3)/2\\(3)\\(4)\end{array}
\left[\begin{array}{cccc|c}
1 & 1 & 0 & 0 & -7/3\\
0 & -1 & 0 & 0 & -103/6\\
0 & 0 & 1 & 0 & 145/12\\
0 & 0 & 0 & 1 & -19/4
\end{array}\right]
\sim
\begin{array}{l}(1)+(2)\\(2)\\(3)\\(4)\end{array}
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 103/6\\
0 & 0 & 1 & 0 & 145/12\\
0 & 0 & 0 & 1 & -19/4
\end{array}\right]$$ nekje je napaka

pravilna rešitev: $$\begin{array}{c}
\begin{array}{cccc} A & B & C & D \end{array}\\
\left[\begin{array}{cccc|c}
1 & & & & -1/2\\
 & 1 & & & -11/6\\
 & & 1 & & 31/12\\
 & & & 1 & 19/4
\end{array}\right]
\end{array}$$

$$\Rightarrow \int \frac{\cdots}{\cdots} = -\frac{1}{2}\int\frac{1}{x-1} - \frac{11}{6}\int\frac{1}{x+1} + \frac{31}{12}\int\frac{1}{x-2} + \frac{19}{4}\int\frac{1}{x+2}$$

$$= -\frac{1}{2}\ln|x-1| - \frac{11}{6}\ln|x+1| + \frac{31}{12}\ln|x-2| + \frac{19}{4}\ln|x+2| + C$$