4.5.2021

5 (d) $\int \cos^5 x\,dx = \int \cos^4 x \cos x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx =$

$t = \sin x,\ dt = \cos x\,dx$

$= \int (1-t^2)^2 dt = \int (1-2t^2+t^4)\,dt = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

(e) $\int \sin^2 x \cos^2 x\,dx = \frac{1}{4}\int (\overbrace{2\sin x\cos x}^{\sin 2x})^2 = \frac{1}{4}\int \sin^2 2x\,dx =$

$= \frac{1}{4}\int \frac{1-\cos 4x}{2}dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$

$\frac{1-\cos 2x}{2} = \sin^2 x$

$\frac{1+\cos 2x}{2} = \cos^2 x$

(f) $\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1+2\cos(2x)+\cos^2(2x))\,dx$

$= \frac{1}{4}\left(x + \sin(2x) + \int \frac{1+\cos 4x}{2}\right) = \frac{1}{4}\left(x + \sin 2x + \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)\right) + C$

6 RECEPT: $p(x,y), q(x,y)$ ... polinoma

- $\int \frac{p(\sin^2 x, \cos^2 x)}{q(\sin^2 x, \cos^2 x)}dx \Longrightarrow \boxed{t = \operatorname{tg} x},\ x = \operatorname{arctg} t,\ dx = \frac{dt}{1+t^2}$

$1 + t^2 = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$\cos^2 x = \frac{1}{1+t^2}$

$\sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{1+t^2}$

$\sin^2 x = \frac{t^2}{1+t^2}$

$= \int \frac{p\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}\right)}{q\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}\right)} \cdot \frac{dt}{1+t^2} = \int \frac{P(t^2)}{Q(t^2)}dt$

- $\int \frac{p(\sin x, \cos x)}{q(\sin x, \cos x)}dx \Longrightarrow t = \operatorname{tg}\frac{x}{2},\ x = 2\operatorname{arctg} t,\ dx = \frac{2\,dt}{1+t^2}$

$- \| - \Longrightarrow \cos^2\frac{x}{2} = \frac{1}{1+t^2}$

$\sin^2\frac{x}{2} = \frac{t^2}{1+t^2}$

$= \int \frac{p\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)}{q\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2} = \int \frac{P(t)}{Q(t)}dt$

$\Longrightarrow \sin x = \sin\left(2\frac{x}{2}\right) = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\sqrt{\frac{t^2}{1+t^2}}\sqrt{\frac{1}{1+t^2}} = \frac{2t}{1+t^2}$

$\Longrightarrow \cos x = \cos\left(2\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$

(a) $\int \frac{dx}{\sin x} = \int \frac{(1+t^2)\,2\,dt}{2t\,(1+t^2)} = \int \frac{dt}{t} = \ln t = \underline{\underline{\ln \operatorname{tg}\frac{x}{2} + C}}$

$t = \operatorname{tg}\frac{x}{2}$

PREIZKUS: $\left(\ln \operatorname{tg}\frac{x}{2} + C\right)' = \frac{1}{\operatorname{tg}\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2} = \frac{\cancel{\cos\frac{x}{2}}}{\sin\frac{x}{2} \cdot \cos^{\cancel{2}}\frac{x}{2} \cdot 2} = \frac{1}{\sin x}$

ALTERNATIVNO (TRIK): $\int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x} = \int \frac{\sin x\,dx}{1-\cos^2 x} = -\int \frac{dt}{1-t^2} = \int \frac{1}{(t-1)(t+1)}$

$t = \cos x,\ dt = -\sin x\,dx$

$= \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) = \frac{1}{2}(\ln|t-1| - \ln|t+1|) = \ln\sqrt{\left|\frac{t-1}{t+1}\right|} + C$

$= \underline{\underline{\ln\sqrt{\left|\frac{\cos x - 1}{\cos x + 1}\right|} + C}}$

(b) $\displaystyle\int \frac{dx}{\sin x+\cos x} = \int \frac{2\,dt/(1+t^2)}{2t/(1+t^2)+(1-t^2)/(1+t^2)} = \int\frac{2\,dt}{1+2t-t^2} = \int \frac{2\,dt}{1-(t-1)^2+1} = \int\frac{dt}{1-\left(\frac{t-1}{\sqrt2}\right)^2} =$

$t=\operatorname{tg}\frac{x}{2}$ $\qquad s=\frac{t-1}{\sqrt2},\ ds=\frac{dt}{\sqrt2}$

$\displaystyle= \int\frac{\sqrt2\, ds}{1-s^2} = -\sqrt2\int\frac{ds}{(s-1)(s+1)} = -\frac{\sqrt2}{2}\int\left(\frac{1}{s-1}-\frac{1}{s+1}\right)$

$\displaystyle= -\frac{\sqrt2}{2}\left(\ln|s-1|-\ln|s+1|\right) = \frac{1}{\sqrt2}\ln\left|\frac{s+1}{s-1}\right| = \frac{1}{\sqrt2}\ln\left|\frac{\frac{t-1}{\sqrt2}+1}{\frac{t-1}{\sqrt2}-1}\right| =$

$\displaystyle= \underline{\underline{\frac{1}{\sqrt2}\ln\left|\frac{\operatorname{tg}\frac x2+\sqrt2-1}{\operatorname{tg}\frac x2-\sqrt2-1}\right|+C}}$

ALTERNATIVNO (TRIK): $\displaystyle\int\frac{1}{\sin x+\cos x}\cdot\frac{\cos x-\sin x}{\cos x-\sin x} = \int\frac{\cos x-\sin x}{\cos^2x-\sin^2x}$

$\displaystyle= \int\frac{\cos x\,dx}{1-2\sin^2x} - \int\frac{\sin x\,dx}{2\cos^2x-1} = \int\frac{ds}{1-2s^2}+\int\frac{dt}{2t^2-1} = \frac{1}{2\sqrt2}\left(-\ln\frac{s-1/\sqrt2}{s+1/\sqrt2}+\ln\frac{t-1/\sqrt2}{t+1/\sqrt2}\right)+C$

$s=\sin x$ $\qquad t=\cos x$

$ds=\cos x\,dx$ $\qquad dt=-\sin x\,dx$ $\qquad \displaystyle=\underline{\underline{\frac{1}{2\sqrt2}\left(-\ln\frac{\sin x-1/\sqrt2}{\sin x+1/\sqrt2}+\ln\frac{\cos x-1/\sqrt2}{\cos x+1/\sqrt2}\right)+C}}$

$\displaystyle\int\frac{dt}{2t^2-1} = \int\frac{1}{(\sqrt2 t-1)(\sqrt2 t+1)} = \frac12\int\left(\frac{1}{\sqrt2t-1}-\frac{1}{\sqrt2t+1}\right) = \frac{1}{2\sqrt2}\int\frac{1}{t-\frac{1}{\sqrt2}}-\frac{1}{t+\frac1{\sqrt2}} = \frac{1}{2\sqrt2}\left(\ln\frac{t-1/\sqrt2}{t+1/\sqrt2}\right)$

$\boxed{7}$ (a) $\displaystyle\int\frac{dx}{5\underbrace{-4x-x^2}} = \int\frac{dx}{5-(x+2)^2+4} = \int\frac{dx}{9-(x+2)^2} = \frac19\int\frac{dx}{1-\left(\frac{x+2}{3}\right)^2} = \frac19\int\frac{3\,dt}{1-t^2} =$

$-(x^2+4x+4)$

$t=\frac{x+2}{3},\ dt=\frac{dx}{3}$

$\displaystyle= \frac{-1}{3}\int\frac{dt}{(t-1)(t+1)} = \frac{-1}{3\cdot2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right) = \frac16\left(\ln|t+1|-\ln|t-1|\right) = \frac16\ln\left|\frac{t+1}{t-1}\right|$

$\displaystyle= \frac16\ln\left|\frac{\frac{x+2}{3}+1}{\frac{x+2}{3}-1}\right| = \frac16\ln\left|\frac{x+5}{x-1}\right|+C$

(b) $\displaystyle\int\frac{dx}{\underbrace{x^2+4x}+9} = \int\frac{dx}{(x+2)^2-4+9} = \frac15\int\frac{dx}{1+\left(\frac{x+2}{\sqrt5}\right)^2} = \frac15\int\frac{\sqrt5\,dt}{1+t^2} = \frac{1}{\sqrt5}\operatorname{arctg} t$

$t=\frac{x+2}{\sqrt5}\Rightarrow dt=\frac{dx}{\sqrt5}$ $\qquad \displaystyle=\underline{\underline{\frac{1}{\sqrt5}\operatorname{arctg}\frac{x+2}{\sqrt5}+C}}$

$\boxed{7.5.2021}$

$\boxed{8}$ $I=\displaystyle\int \frac{x-2\sqrt{1-x^2}}{3x\sqrt{1-x^2}-2(1-x^2)}\,dx$

- $t=\sqrt{1-x^2} \Rightarrow x=\pm\sqrt{1-t^2} \Rightarrow I=\displaystyle\int \frac{\sqrt{1-t^2}-2t}{3\sqrt{1-t^2}\,t-2t^2}\cdot\frac{-t\,dt}{\sqrt{1-t^2}}$

  $dx=\dfrac{-t\,dt}{\sqrt{1-t^2}}$

- $x=\sin t$, $dx=\cos t\,dt \Rightarrow I=\displaystyle\int\frac{(\sin t-2\cos t)\cos t\,dt}{3\sin t\cos t-2\cos^2 t}=\int\frac{\sin t-2\cos t}{3\sin t-2\cos t}\,dt$

$$=\int\frac{\boxed{3\sin t-2\cos t}-2\sin t}{\boxed{3\sin t-2\cos t}}\,dt = t-2\int\frac{\sin t\,dt}{3\sin t-2\cos t}=$$

$\dfrac{3\sin t-2\cos t}{3\sin t-2\cos t}=1$

$s=\operatorname{tg}\frac{t}{2}$, $\sin t=\dfrac{2s}{1+s^2}$, $\cos t=\dfrac{1-s^2}{1+s^2}$

$dt=\dfrac{2\,ds}{1+s^2}$

$$= t-2\int\frac{\frac{2s}{1+s^2}\cdot\frac{2\,ds}{1+s^2}\cdot(1+s^2)}{\left(3\frac{2s}{1+s^2}-2\frac{1-s^2}{1+s^2}\right)\cdot(1+s^2)} = t-2\int\frac{4s\,ds}{(1+s^2)(6s-2+2s^2)}$$

$$=t-\int\frac{4s\,ds}{(1+s^2)(s^2+3s-1)} \qquad s_{1,2}=\frac{-3\pm\sqrt{9-4\cdot(-1)}}{2}=\frac{-3\pm\sqrt{13}}{2}$$

$As^3+(3A+B)s^2+(3B-A)s-B$

$$\frac{4s}{(s^2+1)(s-s_1)(s-s_2)}=\frac{As+B}{s^2+1}+\frac{C}{s-s_1}+\frac{D}{s-s_2}=\frac{(As+B)(s^2+3s-1)+C(s^3-s^2s_2+s-s_2)+D(s^3-s^2s_1+s-s_1)}{-||-}$$

$s^3: 0=A+C+D$

$s^2: 0=3A+B-s_2C-s_1D$

$s: 4=-A+3B+C+D$

$1: 0=-B-s_2C-s_1D$

$$\sim\ \begin{array}{cccc|c} A & B & C & D & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0\\ 3 & 1 & -s_2 & -s_1 & 0\\ -1 & 3 & 1 & 1 & 4\\ 0 & -1 & -s_2 & -s_1 & 0\end{array}\right]\sim\dots\sim$$

$A=\dfrac{8}{3-\sqrt{13}-13}$

$B=\dfrac{4(\sqrt{13}-3)}{3-\sqrt{13}-13}$

$C=D$

$D=\dots$

$$= t-\left(\underbrace{\int\frac{As+B}{s^2+1}}_{B\cdot\operatorname{arctg}s+\frac{A}{2}\int\frac{du}{u}}+C\ln|s-s_1|+D\ln|s-s_2|\right)+E=\underline{\dots x\dots}$$

$B\cdot\operatorname{arctg}s+\dfrac{A}{2}\displaystyle\int\frac{du}{u}=\underline{B\operatorname{arctg}s+\frac{A}{2}\ln|s^2+1|}$

$u=s^2+1$

$du=2s\,ds$