Predavanje 27. december 2021

Vztrajnostni tenzor je simetričen.

Glavne vrednosti, glavne smeri.

Vztrajnostni moment $J_e = \vec{e} \cdot \underline{\underline{J}}\vec{e}$ okoli osi $\vec{e}$, deviacijski momenti $\underline{\underline{D}}$. $= -\int_{\mathcal{B}} \vec{\rho} \otimes \vec{\rho}\, dm$

$$\underline{\underline{J}} = \int_{\mathcal{B}} \left(|\vec{\rho}|^2 \underline{\underline{I}} - \vec{\rho} \otimes \vec{\rho} \right) \underbrace{dm}_{=\rho\, dV}$$

Trditev Tenzor $\underline{\underline{J}}$ je nenegativen. Če telo ni tanka palica, je pozitivno definiten.

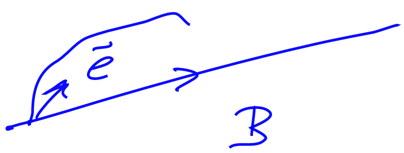
$\lambda_i \geq 0$ $\qquad$ $\lambda_i > 0$

$\vec{e}$ $\qquad$ $\vec{e} \cdot \underline{\underline{J}}\vec{e} \geq 0$ $\qquad$ $\vec{e} \cdot \underline{\underline{J}}\vec{e} = \int_{\mathcal{B}} \left(|\vec{\rho}|^2 - (\vec{e} \cdot \vec{\rho})^2 \right) dm$

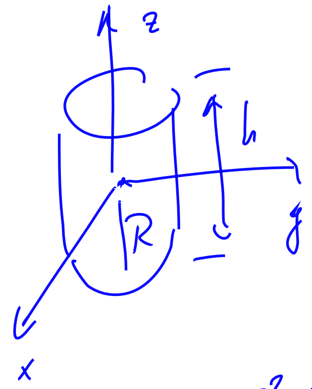
$\vec{e} \cdot (\vec{\rho} \otimes \vec{\rho})\vec{e} = (\vec{e} \cdot \vec{\rho})\vec{\rho} \cdot \vec{e}$ $\qquad$ $|\vec{e}|^2 |\vec{\rho}|^2 - (\vec{e} \cdot \vec{\rho})^2 \geq 0$

$|\vec{a}||\vec{b}| \geq |\vec{a} \cdot \vec{b}|$ $\qquad$ $|\vec{\rho}|^2 > (\vec{e} \cdot \vec{\rho})^2$

enakost velja $\Leftrightarrow$ $\vec{a} \parallel \vec{b}$ $\qquad$ $\vec{e}$ ni vedno vzporeden $\vec{\rho}$



Primer: vztrajnostni tenzor pokončnega valja.



$$V = \pi R^2 \cdot h$$

$$J_{11} = \int_B (y^2 + z^2)\rho_0\, dV \qquad J_{22} = \int_B (x^2 + z^2)\rho_0\, dV$$

$$J_{33} = \int_B (x^2 + y^2)\rho_0\, dV$$

$$K_x = \int_B x^2 \rho_0\, dV = \rho_0 \int_0^R r\,dr \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz\; r^2 \underline{\cos^2\varphi} = \rho_0 \int_0^R r^3\,dr\, \underline{\int_0^{2\pi} \cos^2\varphi\, d\varphi}\; h$$

$$dV = r\,dr\,d\varphi\,dz \qquad = \rho_0 h\, \frac{1}{4} R^4\, \frac{1}{2} \cdot 2\pi = \frac{1}{4}\rho_0 R^4 h \qquad \overset{''}{\int_0^{2\pi}} \sin^2\varphi\, d\varphi$$

$$K_x = \frac{1}{4} m R^2$$

$$K_y = \int_B y^2 \rho_0\, dV = \frac{1}{4} \underline{m} R^2$$

$$K_z = \int_B z^2 \rho_0\, dV = \rho_0 \int_0^R r\,dr \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz\; z^2 = \rho_0\, \frac{1}{2} R^2 \cdot 2\pi\, \underline{2} \int_0^{h/2} z^2\, dz$$

$$= \rho_0 \pi 2 R^2 \cdot \frac{1}{3}\, \frac{h^3}{8} = \frac{1}{12} m h^2$$

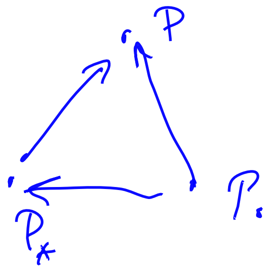
$$J_{11} = \frac{1}{4} m \left(R^2 + \frac{1}{3} h^2\right) = J_{22} \qquad \boxed{J_{33} = \frac{1}{2} m R^2}$$

$$J_{12} = -\int_B x y \rho_0\, dV = -\rho_0 \int_0^R r\,dr \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} dz\; r^2 \cos\varphi \sin\varphi =$$

$$= -\rho_0 h \int_0^R r^3\,dr\, \underline{\int_0^{2\pi} \cos\varphi \sin\varphi\, d\varphi} = 0$$

$$J_{13} = -\int_B x z \rho_0\, dV = J_{23} = 0$$

$\vec{\rho} = P - P_*$



Steinerjev izrek

$$\underline{\underline{J}}(P_*) = \int_B \left(|P-P_*|^2 \underline{\underline{I}} - (P-P_*)\otimes(P-P_*)\right)\rho\, dV$$

$$\vec{r}_* = \int_B \vec{r}\, dm$$

$$\underline{\underline{J}}(P_0) = \int_B \left(|P-P_0|^2 \underline{\underline{I}} - (P-P_0)\otimes(P-P_0)\right)\rho\, dV =$$

$$\int_B (P-P_*)\, dm = \int_B \vec{\rho}\, dm = \vec{0}$$

$$P-P_0 = (P-P_*) + (P_*-P_0)$$

$$|P-P_0|^2 = |P-P_*|^2 + 2(P-P_*)(P_*-P_0) + |P_*-P_0|^2$$

$$(P-P_0)\otimes(P-P_0) = (P-P_*)\otimes(P-P_*) + (P-P_*)\otimes(P_*-P_0) + (P_*-P_0)\otimes(P-P_*) + (P_*-P_0)\otimes(P_*-P_0)$$

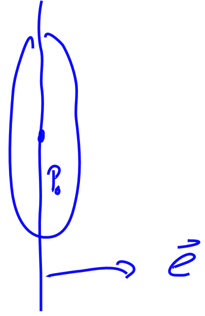
$$\underline{\underline{J}}(P_0) = \underline{\underline{J}}(P_*) + |P_*-P_0|^2 m \underline{\underline{I}} - m(P_*-P_0)\otimes(P_*-P_0)$$

Izrek 1 (Steiner)**.**

$$\underline{\underline{J}}(P_0) = \underline{\underline{J}}(P_*) + m|P_0 - P_*|^2\underline{\underline{I}} - m(P_0 - P_*)\otimes(P_0 - P_*).$$

Trditev 1 (Prehod na novo bazo)**.**

$$\vec{e}\,'i = \alpha_i{}^j\vec{e}_j \Longrightarrow J'_{ij} = \alpha_i{}^k\alpha_j{}^k J_{kl}.$$



Simetrije vztranostnega tenzorja

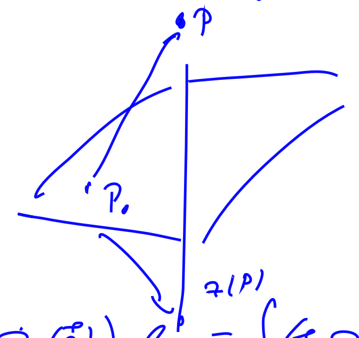
Trditev 2. *Normala na ravnino zrcalne simetrije skozi P_0 je glavna smer vztrajnostnega tenzorja $\underline{\underline{J}}(P_0)$.*

$\underline{\underline{J}}(P_0)\vec e = J_e \vec e$ $\qquad$ $\underline{\underline{J}}(P_0)\vec e \parallel \vec e \iff (\underline{\underline{J}}(P_0)\vec e)\cdot\vec f = 0$

za $\forall\, \vec f \perp \vec e$

$\underline{\underline{J}}(P_0)\vec e = \int_B \left(|\vec\rho|^2 \vec e - (\vec e\cdot\vec\rho)\vec\rho\right)\rho\, dV$

$\left(\int_B (\vec e\cdot\vec\rho)\,\vec\rho\,\rho\, dV\right)\cdot \vec f = 0$ ✓



$\int_B (\vec e\cdot\vec\rho)(\vec f\cdot\vec\rho)\,dm = \int (\vec e\cdot Z(\vec\rho))(\vec f\cdot Z(\vec\rho))\,dm = \int_B (\vec e\cdot Z(\vec\rho))(\vec f\cdot Z(\vec\rho))\,dm$

$Z(B) = B$

zrcaljenje $\qquad Z(P) = P - 2(\vec e\cdot(P-P_0))\,\vec e$; $\vec\rho = P - P_0$

$\vec\rho \mapsto Z(\vec\rho) = Z(P) - P_0$

$(Z(P)-P_0)\cdot\vec e = (P-P_0)\cdot\vec e - 2(\vec e\cdot(P-P_0))\,1 = -(P-P_0)\cdot\vec e = -\vec\rho\cdot\vec e$

$(Z(P)-P_0)\cdot\vec f = (P-P_0)\cdot\vec f - 2(\vec e\cdot(P-P_0))\,\vec e\cdot\vec f = (P-P_0)\cdot\vec f = \vec\rho\cdot\vec f$

→ (**Trditev 3.** *Če ima telo dve ravnini zrcalne simetrije skozi P_0, je presek ravnin simetrij glavna smer vztrajnostnega tenzorja $\underline{\underline{J}}(P_0)$.*)

$= \int_B (-\vec\rho\cdot\vec e)(\vec\rho\cdot\vec f)\,dm = -\int_B (\vec\rho\cdot\vec e)(\vec\rho\cdot\vec f)\,dm \Rightarrow \int_B (\vec\rho\cdot\vec e)(\vec\rho\cdot\vec f) = 0$

Σ_1, Σ_2 ; $\vec e_1, \vec e_2$ $\qquad$ $\underline{\underline{J}}(P_0)\vec e_1 = J_1 \vec e_1$

$\underline{\underline{J}}(P_0)\vec e_2 = J_2 \vec e_2$

$P_0 \in \Sigma_1 \cap \Sigma_2$

$\vec e_3 = \vec e_1 \times \vec e_2$ $\qquad$ $\underline{\underline{J}}(P_0)\vec e_3 \parallel \vec e_3$ ✓

$\vec e_i \cdot \underline{\underline{J}}(P_0)\vec e_3 = \underline{\underline{J}}(P_0)\vec e_i \cdot \vec e_3 = J_i\, \vec e_i\cdot\vec e_3 = J_i\, \vec e_i\cdot(\vec e_1\times\vec e_2)$

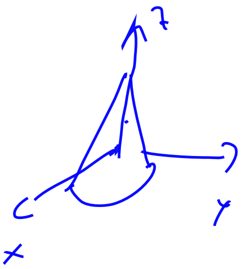
$i = 1, 2$ $\qquad\qquad = 0$

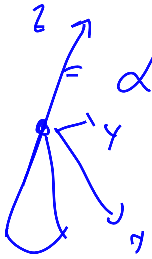
Kinetična energija togega telesa

Kako se komponente vztrajnostnega tenzorja odvisne od baze?

$$\underline{\underline{J}} \to J_{ij} = \vec{e}_i \cdot \underline{\underline{J}}\,\vec{e}_j \qquad \vec{e}_i \text{ baza}$$

$$\vec{e}_i' = \alpha_{ik}\vec{e}_k \qquad J'_{ij} = \vec{e}_i' \cdot \underline{\underline{J}}\,\vec{e}_j' = \alpha_{ik}\vec{e}_k \cdot \underline{\underline{J}}\,\alpha_{jl}\vec{e}_l =$$





$$= \alpha_{ik}\alpha_{jl}\,\vec{e}_k \cdot \underline{\underline{J}}\,\vec{e}_l = \boxed{\alpha_{ik}\alpha_{jl}J_{kl} = J'_{ij}}$$

$$\boxed{T = \frac{1}{2}m|\vec{v}_*|^2 + \frac{1}{2}\vec{\omega}\cdot\underline{\underline{J}}(P_*)\vec{\omega}.}$$

$$T = \int_B \frac{1}{2}\rho|\vec{v}|^2\,dV \qquad \vec{v} = \vec{v}_* + \vec{\omega}\times\vec{\rho} + \underbrace{\cancel{\vec{v}_{rel}}}_{=\vec{0}}$$

$$\vec{v}\cdot\vec{v} = |\vec{v}_*|^2 + 2\vec{v}_*\cdot(\vec{\omega}\times\vec{\rho}) + (\vec{\omega}\times\vec{\rho})\cdot(\vec{\omega}\times\vec{\rho})$$

$$\vec{\rho} = \vec{P} - \vec{P}_*$$

$$2\int_B \vec{v}_*\cdot(\vec{\omega}\times\vec{\rho})\,dm = 2\vec{v}_*\cdot\vec{\omega}\times\underbrace{\int_B\vec{\rho}\,dm}_{=0} = 0$$

$$|\vec{\omega}\times\vec{\rho}|^2 = |\vec{\omega}|^2|\vec{\rho}|^2 - (\vec{\omega}\cdot\vec{\rho})^2$$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

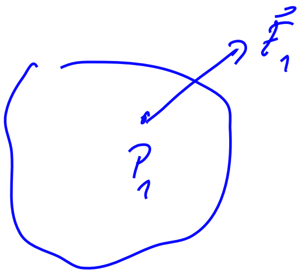
$$\int_B |\vec{\omega}\times\vec{\rho}|^2\,dm = \int_B \left(|\vec{\rho}|^2|\vec{\omega}|^2 - (\vec{\omega}\cdot\vec{\rho})^2\right)dm = \vec{\omega}\cdot\underline{\underline{J}}(P_*)\vec{\omega}$$

$$\vec{\omega}\cdot\underline{\underline{I}}\,\vec{\omega} \qquad \vec{\omega}\cdot(\vec{\rho}\otimes\vec{\rho})\,\vec{\omega}$$

Dinamika togega telesa

Dinamične enačbe togega telesa so:

1. enačba gibanja masnega središča: $m\ddot{\vec{r}}_* = \vec{F}$;
2. izrek o vrtilni količini: $\frac{\mathrm{d}\vec{\mathcal{L}}(P_*)}{\mathrm{d}t} = \vec{N}(P_*)$;
3. $\vec{N}(P_0) = \vec{N}(P_*) + (P_* - P_0) \times \vec{F}$.



$\mathfrak{F} = \{(P_1, \vec F_1), \dots, (P_N, \vec F_N)\}$

$\vec F = \int_B \vec f \, dm$

$\vec f$ gostota volumenske sile

Sistem sil

Sistem sil; točkovne sile, volumenske sile.

$\vec F(\mathfrak F) = \sum_{i=1}^{N} \vec F_i \qquad \vec N(\mathfrak F, O) = \sum_{i=1}^{N} (P_i - O) \times \vec F_i$

$\mathfrak F_1$ in $\mathfrak F_2$ imata enak dinamični vpliv na togo telo, če je $\vec F(\mathfrak F_1) = \vec F(\mathfrak F_2)$ in $\vec N(\mathfrak F_1, O) = \vec N(\mathfrak F_2, O)$

Potem pravimo, da sta sistema sil ekvivalentna, ekvipolentna $\mathfrak F_1 \equiv \mathfrak F_2$

$\vec N(\mathfrak F_1, \hat O) = \vec N(\mathfrak F_1, O) + (O - \hat O) \times \vec F(\mathfrak F_1)$

$$= \vec{N}(\mathcal{F}_2, O) + (O - O') \times \vec{F}(\mathcal{F}_2) = \vec{N}(\mathcal{F}_2, O')$$

Polznost sile

Ekvipolentnost sistema sil.

Dvojica, sistem sil s skupnim prijemališčim, redukcija na skupno prijemališče.

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Trditev 4. *Homogena sila je ekvipolentna svoji rezultanti s prijemališčem v masnem središču.*

Eulerjeva dinamična enačba

Izrek 2 (Eulerjeva dinamična enačba)**.** *Naj bo P_0 fiksna točka ali masno središče. Potem je*

$$\underline{\underline{J}}(P_0)\dot{\vec{\omega}} + \vec{\omega} \times \underline{\underline{J}}(P_0)\vec{\omega} = \vec{N}(P_0).$$

Komponentni zapis Eulerjeve dinamične enačbe.

V lastnem koordinatnem sistemu

$$
\begin{aligned}
N_1 &= J_1\omega_1 - \omega_2\omega_3(J_2 - J_3),\\
N_2 &= J_2\omega_2 - \omega_3\omega_1(J_3 - J_1),\\
N_3 &= J_3\omega_3 - \omega_1\omega_2(J_1 - J_2).
\end{aligned}
$$

.

V koordinatnem sistemu s koordinatno osjo v smeri stalne osi rotacije

$$
\begin{aligned}
N_1 &= J_{13}\ddot{\varphi} - J_{23}\dot{\varphi}^2, \\
N_2 &= J_{23}\ddot{\varphi} + J_{13}\dot{\varphi}^2, \\
N_3 &= J_{33}\ddot{\varphi}
\end{aligned}
$$

Izrek o delu

Trditev 5 (Izrek o delu)**.**

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \vec{v}_* \cdot \vec{F} + \vec{\omega} \cdot \vec{N}_*.$$

Izrek 3 (Izrek o energiji)**.** *Če je sistem sil na togo telo konzervativen, je vsota kinetične in potencialne energije konstanta gibanja.*