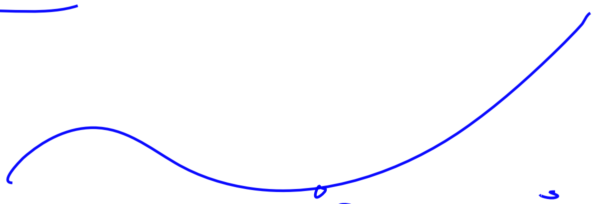
Predavanje 15. november 2021

0.1 Gibanje po krivulj*i



P

$$m\vec{a} = \vec{F} + \vec{S}$$

$$\vec{S} \perp \vec{e}_t \qquad \int_{t_1}^{t_2} \vec{S} \cdot \vec{v}\, dt \qquad \vec{v} \parallel \vec{e}_t$$

$$\int_{P_1}^{P_2} \vec{S} \cdot d\vec{r} = 0$$

Sila vezi, idealna vez ($\vec{S} \cdot \vec{e}_t = 0$).

Trditev 1. *Delo idealne sile vezi je enako nič.*

Redukcija na premočrtno gibanje

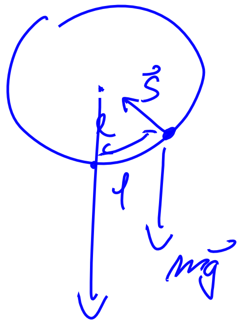
$$m\vec{a} = \vec{F} + \vec{S} \quad / \cdot \vec{e}_t \quad \Rightarrow \quad m\ddot{s} = \vec{F} \cdot \vec{e}_t \quad / \dot{s}$$

$$\frac{1}{2} m \dot{s}^2 + V(s) = E_0$$

$$s = \int_{t_1}^{t} |\vec{v}|\, dt = \int_{t_1}^{t} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

Gibanje po gladki krivulji pod vplivom konzervativne sile je integrabilno.

Matematično nihalo = gibanje po gladki krožnici pod vplivom sile teže



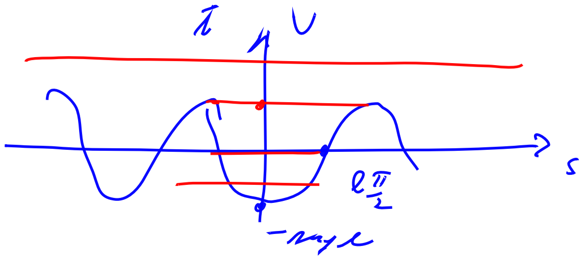
$U = -m\vec g\cdot\vec r$

$\vec g = g\vec i$

$\vec r = l(\cos\varphi\,\vec i + \sin\varphi\,\vec j)$

$U = -mgl\cos\varphi \qquad s = l\varphi$

$U = -mgl\cos\left(\frac{s}{l}\right)$



$$T = \sqrt{2m}\int_{-s_0}^{s_0}\frac{ds}{\sqrt{E_0 + mgl\cos\frac{s}{l}}}$$

$E_0 = -mgl\cos\frac{s_0}{l} \qquad s = \varphi l$

$\cos\varphi = \cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}$

$1 = \cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2}$

$1-\cos\varphi = 2\sin^2\frac{\varphi}{2}$

- aproksimacija majhnega nihanja;
- harmonična aproksimacija;
- libracijska $T = 2\pi\sqrt{l/g}\sqrt{\theta_0/\sin\theta_0}$.
- ocena nihajnega časa matematičnega nihala:

$$2\pi\sqrt{\frac{l}{g}} \le T \le \frac{1}{\cos\frac{\theta_0}{2}}2\pi\sqrt{\frac{l}{g}}.$$

$$T = \sqrt{2m}\frac{1}{\sqrt{mgl}}\int_{-\varphi_0}^{\varphi_0}\frac{l\,d\varphi}{\sqrt{\cos\varphi - \cos\varphi_0}} = 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\cos\varphi-\cos\varphi_0}} =$$

$$2\sqrt{\frac{l}{g}}\int_0^{\varphi_0}\frac{d\varphi}{\sqrt{\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi}{2}}} = 4\sqrt{\frac{l}{g}}\int_0^1\frac{\sin\frac{\varphi_0}{2}\,du}{\sqrt{1-\sin^2\frac{\varphi_0}{2}u^2}\;\sin\frac{\varphi_0}{2}\sqrt{1-u^2}}$$

$\sin\frac{\varphi}{2} = \sin\frac{\varphi_0}{2}\cdot u$

$$\frac{1}{2}\cos\frac{\varphi}{2}\,d\varphi = \sin\frac{\varphi_0}{2}\,du \Rightarrow d\varphi = 2\sin\frac{\varphi_0}{2}\,du\,\frac{1}{\sqrt{1-\sin^2\frac{\varphi_0}{2}u^2}}$$

$$T = 4\sqrt{\frac{l}{g}}\int_0^1\frac{du}{\sqrt{(1-\sin^2\frac{\varphi_0}{2}u^2)(1-u^2)}}$$

Popolni eliptični integral 1. stopnje

$$1 \le \frac{1}{\sqrt{1-\sin^2\frac{\varphi_0}{2}u^2}} \le \frac{1}{\cos\frac{\varphi_0}{2}}$$

$$2\pi\sqrt{\frac{l}{g}} = 4\sqrt{\frac{l}{g}}\int_0^1\frac{du}{\sqrt{1-u^2}} \le T \le \frac{1}{\cos\frac{\varphi_0}{2}}2\pi\sqrt{\frac{l}{g}}$$

" $\arcsin 1$"

$$T_0 = 2\pi\sqrt{\frac{m}{U''(s=0)}} = 2\pi\sqrt{\frac{ml}{mg}} \doteq 2\pi\sqrt{\frac{l}{g}} \qquad (\sin\varphi \doteq \varphi)$$

$$U = -mgl\cos\frac{s}{l}; \quad U' = mg\sin\frac{s}{l}; \quad U'' = \frac{mg}{l}\cos\frac{s}{l}$$

Harmonična aproksimacija periode gibanja po krivulji. Libracijska aproksimacija

$$T = 2\pi\sqrt{1/gf''(x_0)}.$$

$$T = \sqrt{\frac{m}{2}}\,\pi\left(\frac{1}{\sqrt{\chi(a)}} + \frac{1}{\sqrt{\chi(b)}}\right) \qquad E_0 - U = (b-x)(x-a)\chi(x)$$

$$(s_0-s)(s+s_0) = s_0^2 - s^2 \qquad b = s_0,\quad a = -s_0,\quad x = s$$

$$\chi(s) = \frac{E_0 + mgl\cos\left(\frac{s}{l}\right)}{s_0^2 - s^2}$$

$$T = \sqrt{2m}\,\pi\frac{1}{\sqrt{\chi(s_0)}} \qquad \chi(s_0) = \lim_{s\to s_0}\frac{-mg\sin\frac{s}{l}}{-2s} = \frac{1}{2}mg\frac{\sin\frac{s_0}{l}}{s_0} = \frac{mg\sin\varphi_0}{2\varphi_0 l}$$

$$T = 2\pi\sqrt{\frac{l\varphi_0}{g\sin\varphi_0}}$$

$$T \doteq 2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{\varphi_0}{\sin\varphi_0}}$$

Gibanje v polju centralne sile

$$T = 2\pi\sqrt{\frac{m}{U''(s_0)}} \qquad \vec{e}_t$$

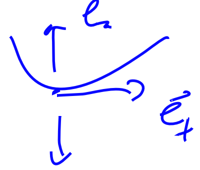
$$U(s) = U(\vec{r}(s))$$

$$U' = \operatorname{grad}U\cdot\frac{d\vec r}{ds} \qquad U'' = \left(\frac{\partial^2 U}{\partial \vec r^2}\cdot\frac{d\vec r}{ds}\right)\cdot\frac{d\vec r}{ds} + \operatorname{grad}U\,\frac{d^2\vec r}{ds^2} \qquad \varkappa\cdot\vec e_n$$

$$U = -m\vec g\cdot\vec r; \quad \operatorname{grad}U = -m\vec g; \quad U'' = -m\vec g\cdot\varkappa\,\vec e_n; \quad -m\vec g\cdot\vec e_t = 0$$

Definicija 1. *Sila $\vec{F}$ je centralna, če obstaja točka O (center sile) tako, da velja $\vec F = f(|P-O|)(P-O)/|P-O|$.*

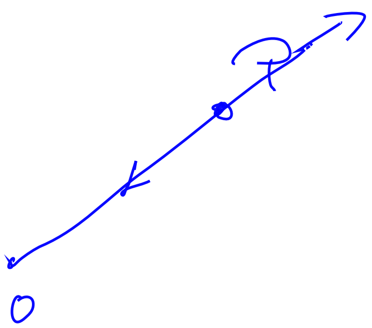
Ravninska krivulja $\Rightarrow \vec g \parallel \vec e_n$ $\qquad U'' = +mg\varkappa$



$$\vec g = -g\,\vec e_n \qquad \boxed{T \doteq 2\pi\sqrt{\frac{1}{g\varkappa}}}$$

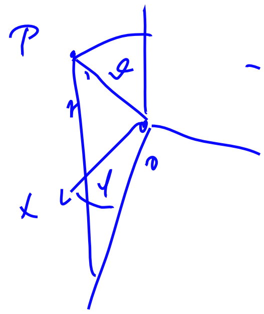
Krožnica $\quad \varkappa = \frac{1}{l} \quad T \doteq 2\pi\sqrt{\frac{l}{g}}$

$r = l$



$$\vec F = F(|P-O|)\frac{P-O}{|P-O|} = F(r)\,\vec e_r$$

$$P - O = \overrightarrow{OP} = \vec r$$



$$-\vec F = \operatorname{grad} U = \frac{\partial U}{\partial r}\vec e_r + \underbrace{\frac{1}{r}\frac{\partial U}{\partial \vartheta}\vec e_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial U}{\partial \varphi}\vec e_\varphi}$$

$\vec F$ centralna $\qquad \frac{\partial U}{\partial \vartheta} = \frac{\partial U}{\partial \varphi} = 0$

$$\vec F = -\frac{dU}{dr}\vec e_r$$

$$U = -\int_{x_0}^{x} f(\xi)\,d\xi + U(x_0)$$

$$\boxed{\vec F = f(r)\vec e_r} \; ; \qquad U(r) = -\int_{r_0}^{r} f(r)\,dr + U(r_0) \; ; \qquad -\frac{dU}{dr} = f(r)$$

Če je funkcija $f = f(r)$ zvezna, je centralna sila konzervativna.

Energija je konstanta gibanja v polju centralne sile.

Vrtilna količina

$$\vec l = \vec l(O,P) = (P-O)\times m\vec v = \vec r\times m\vec v$$

$$\dot{\vec l} = \dot{\vec r}\times m\vec v + \vec r\times m\vec a = \vec r\times\vec F = \vec r\times f(r)\vec e_r = \vec 0.$$

Trditev 2. *Pri gibanju v polju centralnih sil je vrtilna količina $\vec l(O,P)$ konstantna.*

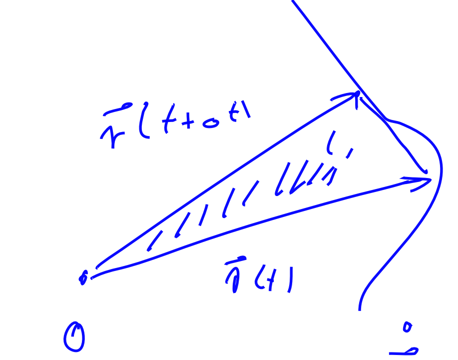
integral gibanja

Trditev 3. *Gibanje v polju centralnih sil je ravninsko. Točka se giblje po ravnini, ki vsebuje center sil in ima normalo v smeri vektorja vrtilne količine.*

$$(\vec r - \vec r_0)\cdot\vec n = 0 \qquad\qquad \vec r\cdot\vec l = 0$$

$$0 = \vec r\cdot(\vec r\times m\vec v) = \vec r\cdot(\underbrace{\vec r_0\times m\vec v_0}_{\vec l_0}) = \vec r\cdot\vec l_0$$

$$\vec r_0 = \vec 0$$



$$\Delta\vec{A} = \frac{1}{2}\vec{r}(t)\times\vec{r}(t+\Delta t) =$$

$$= \frac{1}{2}\vec{r}(t)\times\left(\vec{r}(t+\Delta t)-\vec{r}(t)\right)$$

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\vec{r}(t)\times\frac{\Delta\vec{r}}{\Delta t}$$

Ploščinska hitrost

$$\dot{\vec{A}} = \lim_{\Delta t\to 0}\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2}\vec{r}(t)\times\vec{v}(t)$$

$$\vec{l} = \vec{r}\times m\vec{v} = 2m\dot{\vec{A}}$$

Ploščinska hitrost, zveza $l = \left|\vec{l}\right| = mC_0$, kjer je C_0 dvojna ploščinska hitrost.

$$\vec{r}(t)\times\vec{v}(t) = r\vec{e}_r\times\left(\dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi\right) = \underbrace{r^2\dot{\varphi}}\,\vec{e}_r\times\vec{e}_\varphi = r^2\dot{\varphi}\vec{k}$$

$$\vec{l} = mr^2\dot{\varphi}\vec{k} \;;\quad \dot{\vec{A}} = \frac{1}{2}r^2\dot{\varphi}\vec{k}$$

$r^2\dot{\varphi} = C_0$ dvojna ploščinska hitrost

Izrek 1. *Gibanje v polju centralne sile ima dve konstanti gibanja, energijo in ploščinsko hitrost.*

$|\vec{l}| = l \;;\quad l = mC_0$ $E_0,\ C_0$

Keplerjevi zakoni

Zgodovinski oris: Tycho Brahe (1546-1601), Johannes Kepler (1571-1630), Isaac Newton (1643-1727);

Trditev 4. *Če je ploščinska hitrost konstantna, je obodni pospešek enak nič.*

Binetova formula

$$a_r = -C_0^2u^2\left(d^2u/d\theta^2 + u\right).$$

Trditev 5. *Če velja prvi Keplerjev zakon, je radialni pospešek obratno sorazmeren kvadratu oddaljenosti do centra sil.*