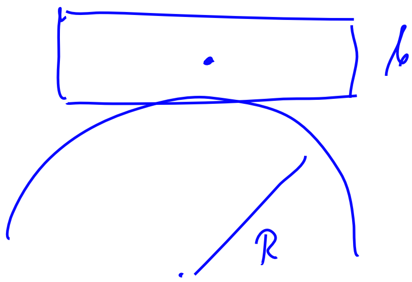
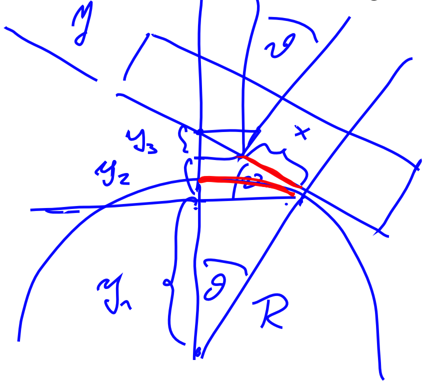
Vaje 16. november 2021



1. Na vrh krožnice s polmerom R_0 postavimo pravokotnik z višino b. Določi največjo višino b, da bo ravnovesna lega štabilna.



$$y_1 = R\cos\vartheta$$

$$y_2 = x \cancel{\cos\vartheta}\ \sin\vartheta \qquad x = R\vartheta$$

$$y_3 = \frac{b}{2}\cos\vartheta$$

$$V = mgy = mg\left[\cos\vartheta\left(R + \cancel{R\vartheta} + \frac{b}{2}\right) + R\vartheta\sin\vartheta\right]$$

$$\frac{dV}{d\vartheta} = -mg\sin\vartheta\left(R + \frac{b}{2}\right) + mgR\sin\vartheta + mgR\vartheta\cos\vartheta$$

$$\frac{dV}{d\vartheta} = 0 \qquad \frac{dV}{d\vartheta}(\vartheta = 0) = 0$$

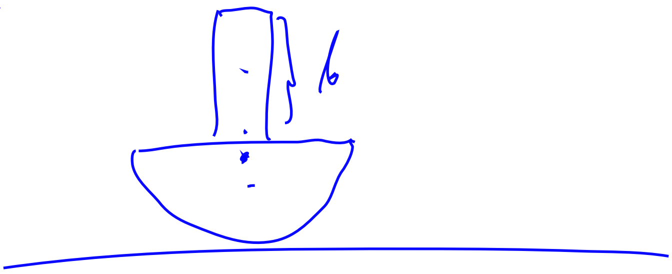
$$V'' = mg\left[-\cos\vartheta\left(R + \frac{b}{2}\right) + R\cos\vartheta + R\cos\vartheta - R\vartheta\sin\vartheta\right]$$

$$V''(\vartheta = 0) = mg\left[-R - \frac{b}{2} + 2R\right] = mg\left(R - \frac{b}{2}\right)$$

Pogoj stabilnosti $\quad V''(\vartheta = 0) > 0 \iff R > \frac{b}{2}$

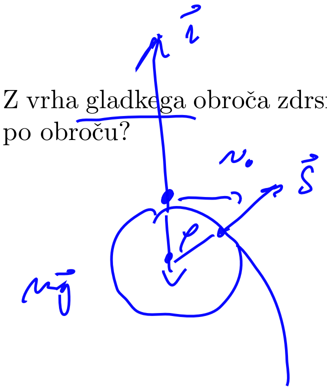
$$\boxed{b < 2R}$$

$$\cancel{T = 2\pi\sqrt{\frac{m}{V''}}}$$



$\vec{e}_r = \cos\varphi\, \vec{i} + \sin\varphi\, \vec{j}$

2. Z vrha gladkega obroča zdrsne materilna točka z obodno brzino v_0. Kje točka zapusti gibanje po obroču?



$m\vec{a} = m\vec{g} + \vec{S}$ $\quad \vec{S} = S\vec{e}_r$

$S \geq 0$

točka zapusti obroč pri $S = 0$.

$$m(-R\dot\varphi^2 \vec{e}_r + R\ddot\varphi\, \vec{e}_\varphi) = -mg\vec{i} + S\vec{e}_r \quad /\cdot \vec{e}_r \quad / \vec{e}_\varphi$$

$$-mR\dot\varphi^2 = -mg\,\vec{i}\cdot\vec{e}_r + S = -mg\cos\varphi + S$$

$$\to \quad mR\ddot\varphi = -mg\,\vec{i}\cdot\vec{e}_\varphi = mg\sin\varphi \quad /\dot\varphi$$

$$R\dot\varphi\ddot\varphi = g\dot\varphi\sin\varphi \Rightarrow \left(\tfrac12 R\dot\varphi^2\right)^{\cdot} = (-\cos\varphi)^{\cdot}\, g$$

$$\tfrac12 R\dot\varphi^2 + g\cos\varphi = E_0 = \tfrac{1}{2R} v_0^2 + g \qquad v_0 = R\dot\varphi_0$$

$$R\dot\varphi^2 = \frac1R v_0^2 + 2g - 2g\cos\varphi$$

$S = 0$ $$R\dot\varphi^2 = g\cos\varphi = \frac1R v_0^2 + 2g - 2g\cos\varphi$$

$$3g\cos\varphi = \frac1R v_0^2 + 2g$$

$$\cos\varphi = \frac{1}{3g}\left(2g + \frac1R v_0^2\right) = \frac23 + \frac{v_0^2}{3gR} \leq 1$$

$$\frac{v_0^2}{3gR} \leq \frac13 \;; \quad v_0^2 < gR \;; \quad v_0 < \sqrt{gR}$$

$v_0 \geq \sqrt{gR}$ točka takoj zapusti obroč $\quad \Big|\quad \varphi = \arccos\left(\frac23 + \frac{v_0^2}{3gR}\right)$

$$y = v_0 t$$

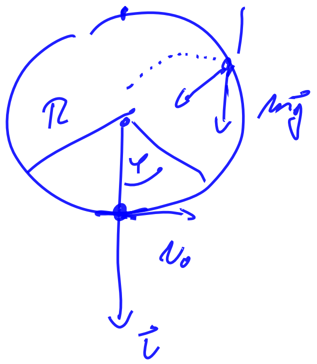
$$x = R - \tfrac12 g t^2 = R - \tfrac12 g \frac{1}{v_0^2} y^2$$

$$\varkappa = \left|\frac{x''}{(x'^2 + y'^2)^{3/2}}\right| = \left|\frac{-\frac{g}{v_0^2}}{(0+1)^{3/2}}\right| = \frac{g}{v_0^2} \qquad x' = -\frac{g}{v_0^2} y$$

$\varkappa < \frac1R$ pogoj, da točka takoj zapusti parabolo

$$\frac{g}{v_0^2} < \frac1R \qquad \boxed{gR < v_0^2}$$

3. Po notranjosti vertikalno postavljenega gladkega obroča se giblje materialna točka pod vplivom sile teže. Določi minimalno brzino na dnu obroča, da bo točka zakrožila. .



$$\frac{1}{2} m v_0^2 = mg2R$$

$$m\vec{a} = m\vec{g} + \vec{S} \qquad \vec{S} = -S\vec{e}_r$$

$$S \geq 0$$

$$m\left(-R\dot{\varphi}^2 \vec{e}_r + R\ddot{\varphi}\, \vec{e}_\varphi\right) = mg\vec{i} - S\vec{e}_r \quad / \cdot \vec{e}_r \quad / \vec{e}_\varphi$$

$$-mR\dot{\varphi}^2 = mg\cos\varphi - S$$

$$\cancel{m}R\ddot{\varphi} = -\cancel{m}g\sin\varphi \;/ \dot{\varphi} \Rightarrow \frac{1}{2}R(\dot{\varphi}^2)^{\cdot} = (g\cos\varphi)^{\cdot}$$

$$\frac{1}{2}R\dot{\varphi}^2 - g\cos\varphi = E_0 = \frac{1}{2R}v_0^2 - g$$

$$R\dot{\varphi}^2 = \frac{1}{R}v_0^2 + 2g\cos\varphi - 2g \Rightarrow \varphi = \varphi(t)$$

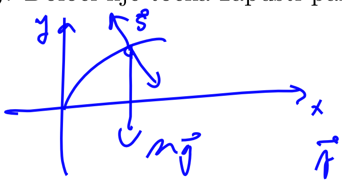
$$S = mg\cos\varphi + mR\dot{\varphi}^2 = m\left[g\cos\varphi + \frac{1}{R}v_0^2 + 2g\cos\varphi - 2g\right]$$

$$S \geq 0 \qquad 3g\cos\varphi + \frac{1}{R}v_0^2 - 2g \geq 0 \qquad \varphi \in [0, \pi]$$

$$\varphi = \pi \Rightarrow -5g + \frac{1}{R}v_0^2 \geq 0 \Rightarrow v_0^2 \geq 5gR \qquad v_0 \geq \sqrt{5gR}$$

4. Točka se giblje brez trenja po paraboli $y^2 = 2px$ pod vplivom sile teže v negativni smeri osi y. Določi kje točka zapusti parabolo, če točka prične gibanje iz mirovanja pri $x = 2p$. .

$y = \sqrt{2px}$



$$m\vec a = m\vec g + \vec S$$

$$\vec r = x\vec i + \sqrt{2px}\,\vec j = \frac{1}{2p} y^2 \vec i + y\vec j$$

$$\dot{\vec r} = \frac{d\vec r}{dy}\dot y = \dot y\left(\frac1p y\,\vec i + \vec j\right)$$

$$\vec a = \ddot{\vec r} = \ddot y\left(\frac1p y\,\vec i + \vec j\right) + \dot y^2\left(\frac1p \vec i\right) = \frac1p\left(\dot y^2 + y\ddot y\right)\vec i + \ddot y\,\vec j$$

$$m\left(\frac1p(\dot y^2+y\ddot y)\vec i + \ddot y\,\vec j\right) = -mg\vec j - S\vec e_n \quad \Big/\left(\frac1p y\vec i+\vec j\right) \quad \Big/\ \vec i - \frac1p y\vec j$$

$$\frac{1}{p^2}y(\dot y^2+y\ddot y)+\ddot y = -g \quad — \quad \dot y$$

$\frac1p y\vec i + \vec j \parallel \vec e_t$

$\vec i - \frac1p y \vec j \parallel \vec e_n$

$S=0$

$$\frac1p(\dot y^2 + y\ddot y) - \frac1p y\ddot y = +\frac1p g y$$

$$\frac1p\dot y^2 = \frac1p g y \qquad \boxed{\dot y^2 = gy}$$

Na začetku

$\dot y = 0,\ y = y_0 = 2p$

$E_0 = gy_0$

$$\dot y\ddot y + \frac{1}{p^2}\left(y\dot y^3 + y^2\dot y\ddot y\right) = -g\dot y$$

$$\left(\frac12\dot y^2\right)^{\cdot} + \frac{1}{p^2}\,\frac12\left(y^2\dot y^2\right)^{\cdot} = (-gy)^{\cdot} \Rightarrow \frac12\dot y^2 + \frac{1}{2p^2}y^2\dot y^2 + gy = E_0$$

$$\frac12\left(2y\dot y^3 + y^2\cdot 2\dot y\ddot y\right)$$

$$\frac12\dot y^2\left(1+y^2\frac{1}{p^2}\right) = E_0 - gy = g(2p-y)$$

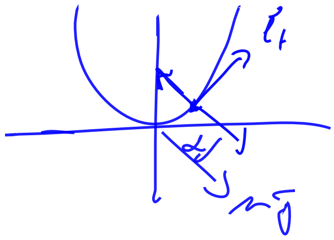
$$\dot y^2 = \frac{2g(2p-y)}{1+y^2/p^2}$$

$$\frac{2g(2p-y)}{1+y^2/p^2} = gy \Rightarrow 2(2p-y) = y + \frac{y^3}{p^2} \Rightarrow \frac{y^3}{p^2}+3y = 4p$$

$$\left(\frac yp\right)^3 + 3\left(\frac yp\right) = 4$$

$$\Downarrow$$

$$y = p, \quad x = \frac p2$$



$y=\frac{1}{2}px^2 \qquad p>0$

brez trenja

5. Točka se giblje paraboli $y=\frac{1}{2}px^2$ pod vplivom sile teže $m\vec{g}=mg(\cos\alpha\vec{\imath}+\sin\alpha\vec{\jmath})$. Določi periodo majhnega nihanja okoli ravnovesne lege.

$$T=2\pi\sqrt{\frac{m}{U''(s_0)}} \qquad U=-m\vec g\cdot\vec r$$

$$\frac{dU}{ds}=-m\vec g\cdot\frac{d\vec r}{ds}=-m\vec g\cdot\vec e_t \qquad \frac{dU}{ds}=0 \Rightarrow \vec g\perp\vec e_t$$

$$(\cos\alpha\vec\imath-\sin\alpha\vec\jmath)\cdot\vec e_t=0$$

$$U''=-m\vec g\cdot\frac{d\vec e_t}{ds}=-m\vec g\varkappa\cdot\vec e_n= \qquad \vec e_t=\sin\alpha\vec\imath+\cos\alpha\vec\jmath$$

$$=+mg\varkappa \qquad \vec e_n=-\cos\alpha\vec\imath+\sin\alpha\vec\jmath$$

$$\vec g=-g\vec e_n$$

$$\boxed{T=2\pi\sqrt{\frac{1}{g\varkappa}}}$$

$$\varkappa=\left|\frac{y''}{(1+y'^2)^{3/2}}\right|=\frac{p}{(1+p^2x^2)^{3/2}}$$

$$y=\frac{1}{2}px^2\ ;\quad y'=px\ ,\quad y''=p \qquad \vec r=x\vec\imath+\frac{1}{2}px^2\vec\jmath$$

$$0=\vec g\cdot\vec e_t \qquad \vec e_t\parallel(\vec\imath+px\vec\jmath)$$

$$0=(\cos\alpha\vec\imath-\sin\alpha\vec\jmath)\cdot(\vec\imath+px\vec\jmath)=\cos\alpha-\sin\alpha\,px$$

$$px=\frac{\cos\alpha}{\sin\alpha}$$

$$\varkappa=\frac{p}{\left(1+\frac{\cos^2\alpha}{\sin^2\alpha}\right)^{3/2}}=\frac{p}{\left(\frac{1}{\sin^2\alpha}\right)^{3/2}}=p\sin^3\alpha$$

$$T=2\pi\frac{1}{\sqrt{gp\sin^3\alpha}}=\frac{2\pi}{\sin\alpha}\frac{1}{\sqrt{gp\sin\alpha}}$$

$$\alpha=\frac{\pi}{2}$$

$$T=\frac{2\pi}{\sqrt{gp}}$$

6. Obravnavaj vsiljeno dušeno nihanje.