Predavanje 22. november 2021

Keplerjevi zakoni

Zgodovinski oris: Tycho Brahe (1546-1601), Johannes Kepler (1571-1630), Isaac Newton (1643-1727);

Trditev 1. *Če je ploščinska hitrost konstantna, je obodni pospešek enak nič.*

$$a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi} \quad / r$$

$$r a_\varphi = r^2\ddot{\varphi} + 2\dot{r} r\dot{\varphi} = (r^2\dot{\varphi})^{\cdot}$$

$$r a_\varphi = (C_0)' = 0$$

Binetova formula

$$\boxed{a_r = -C_0^2 u^2 \left(d^2u/d\theta^2 + u\right).}$$

$r = r(\varphi)$; $\varphi(t)$ $\qquad C_0 = r^2\dot{\varphi}$ $\qquad u = \frac{1}{r}$

$$\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} = \frac{dr}{d\varphi}\cdot\frac{C_0}{r^2} = -C_0\frac{d}{d\varphi}\left(\frac{1}{r}\right) = -C_0\frac{du}{d\varphi} = -C_0 u'$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\varphi}\dot{\varphi} = -C_0 u''\cdot\frac{C_0}{r^2} = -C_0^2 u^2 u''$$

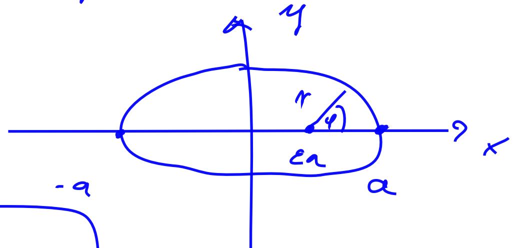
$$a_r = \ddot{r} - r\dot{\varphi}^2 = -C_0^2 u^2 u'' - \frac{1}{u}\cdot C_0^2 u^4 = -C_0^2 u^2 (u'' + u)$$

$u = u(\varphi) \Rightarrow a_r = \dots$

Polarni zapis elipse

Kartezični zapis

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$\boxed{r = \frac{p}{1+\varepsilon\cos\varphi}}$$

$\varepsilon = 0$; $r = p$ krožnica

$0 \le \varepsilon < 1$ elipsa

$\varepsilon = 1$ parabola

$\varepsilon > 1$ hiperbola

Trditev 2. *Če velja prvi Keplerjev zakon, je radialni pospešek obratno sorazmeren kvadratu oddaljenosti do centra sil.*

$$a_r = -C_0^2 u^2 (u + u'')$$

$$u = \frac{1}{r} = \frac{1}{p}(1+\varepsilon\cos\varphi) \qquad u' = -\frac{\varepsilon}{p}\sin\varphi \qquad u'' = -\frac{\varepsilon}{p}\cos\varphi$$

$$a_r = -C_0^2 \frac{1}{r^2}\left(\frac{1}{p}(1+\varepsilon\cos\varphi) - \frac{\varepsilon}{p}\cos\varphi\right) = -\frac{C_0^2}{p}\frac{1}{r^2}$$

$$m\vec{a} = m a_r \vec{e}_r = -m\frac{C_0^2}{p}\frac{1}{r^2}\vec{e}_r = \vec{F} = -\kappa\frac{mM}{r^2}\vec{e}_r$$

$$\frac{C_0^2}{p} = \kappa M$$

Tretji Keplerjev zakon

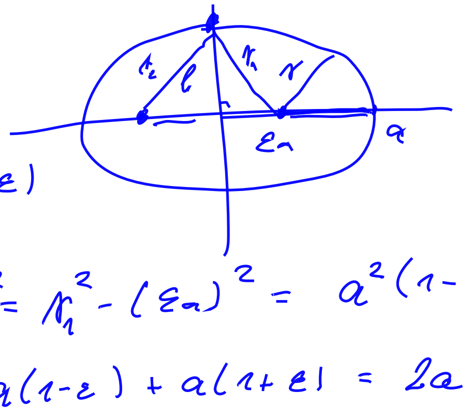
Kvadrat periode je sorazmeren kubu večje polosi elipse.

Trditev: Za vse planete v osončju je kvocient $\frac{C_0^2}{p}$ konstanten.

$$T^2 = k a^3 \qquad \frac{1}{2}C_0 \cdot T = A = \pi a b \Rightarrow T = \frac{2\pi a b}{C_0}$$

ploščina elipse

$$\frac{4\pi^2 a^2 b^2}{C_0^2} = k a^3 \qquad \boxed{\frac{4\pi^2 b^2}{C_0^2} = k a}$$



$$r = \frac{p}{1+\varepsilon\cos\varphi} \qquad \varphi = 0 \Rightarrow \frac{p}{1+\varepsilon} = a(1-\varepsilon)$$

$$a = \frac{p}{1-\varepsilon^2}$$

$$b = a\sqrt{1-\varepsilon^2} = \frac{p}{\sqrt{1-\varepsilon^2}}$$

$$b^2 = r_1^2 - (\varepsilon a)^2 = a^2(1-\varepsilon^2)$$

$$r_1 + r_2 = \text{konst} = a(1-\varepsilon) + a(1+\varepsilon) = 2a$$

$$\frac{4\pi^2 p^2}{C_0^2 \cancel{(1-\varepsilon^2)}} = K \cdot \frac{\cancel{p}}{\cancel{1-\varepsilon^2}} \Rightarrow \frac{C_0^2}{p} = \frac{4\pi^2}{K}$$

Redukcija na premočrtno gibanje

$$\frac{1}{2} m v^2 + V(r) = E_0 \qquad \vec{v} = \dot{r}\,\vec{e}_r + r\dot{\varphi}\,\vec{e}_\varphi$$

$$\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 + V(r) = E_0 \qquad r^2\dot{\varphi} = C_0$$

$$\underbrace{\frac{1}{2} m \dot{r}^2}_{} + \underbrace{\frac{1}{2} m \frac{1}{r^2} C_0^2 + V(r)}_{} = E_0 \qquad l = m C_0$$

$$\frac{1}{2} m \dot{r}^2 + U(r) = E_0 \; ; \quad U = V(r) + \frac{m C_0^2}{2r^2} = V(r) + \frac{l^2}{2mr^2}$$

U efektivni potencial.

Efektivni potencial $U(r) = \frac{l^2}{2mr^2} + V(r)$.

$$t - t_0 = \operatorname{sgn} \dot{r} \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}(E_0 - U(r))}}$$

Določitev trajektorije

$$t = t(r) \Rightarrow r = r(t)$$

$$\dot{\varphi} = \frac{C_0}{r^2} \Rightarrow \varphi - \varphi_0 = \int_{t_0}^{t} \dot{\varphi}\, dt = \int_{t_0}^{t} \frac{C_0}{r^2(t)}\, dt \Rightarrow \varphi = \varphi(t)$$

Določitev tira

$$\theta = \theta_0 + \operatorname{sgn} \dot{r} \frac{l}{\sqrt{2m}} \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{E_0 - U(r)}}.$$

$r = r(\varphi)$

$C_0 = r^2 \dot{\varphi}$

$$\frac{dr}{d\varphi} = \frac{dr}{dt}\frac{dt}{d\varphi} = \dot{r}\frac{1}{\dot{\varphi}} = \dot{r}\frac{r^2}{C_0}$$

$$\frac{dr}{d\varphi} = \frac{1}{C_0} r^2 \operatorname{sgn} \dot{r} \sqrt{\frac{2}{m}(E_0 - U(r))}$$

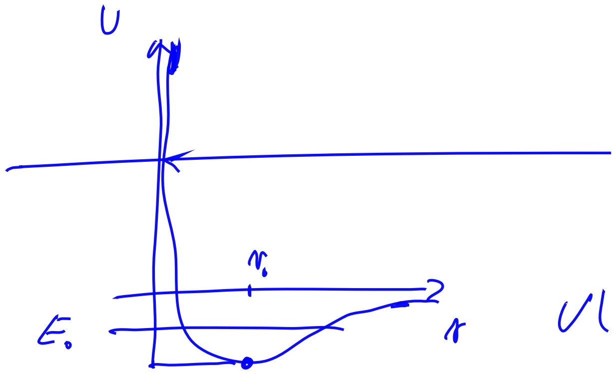
$$\int \frac{C_0 \, dr}{r^2 \sqrt{\frac{2}{m}(E_0 - U(r))}} = \int d\varphi$$

$$\varphi - \varphi_0 = \operatorname{sgn} \dot{r} \frac{l}{m} \sqrt{\frac{m}{2}} \int \frac{dr}{r^2 \sqrt{E_0 - U(r)}} = \operatorname{sgn} \dot{r} \frac{l}{\sqrt{2m}} \int \frac{dr}{r^2 \sqrt{E_0 - U(r)}}$$

$$V(r) = -\frac{\gamma}{r} \qquad \gamma = \kappa m M > 0$$

Gibanje v polju gravitacijske sile

Kvalitativna analiza.



$$U = \frac{l^2}{2mr^2} + V(r) = \frac{l^2}{2mr^2} - \frac{\gamma}{r}$$

$$U' = -\frac{l^2}{mr^3} + \frac{\gamma}{r^2}$$

$$U' = 0 \Rightarrow r_0 = \frac{l^2}{m\gamma} \qquad C_0 = r^2\dot{\varphi}$$

$$U(r_0) = \frac{l^2 m^2 \gamma^2}{2m\, l^4} - \frac{\gamma m \gamma}{l^2} = -\frac{1}{2}\frac{m\gamma^2}{l^2}$$

$$\boxed{-\frac{1}{2}\frac{m\gamma^2}{l^2} < E_0 < 0}$$

$$\varphi - \varphi_0 = \frac{l}{\sqrt{2m}} \int_{r_0}^{r} \frac{dr}{r^2\sqrt{E_0 - \frac{l^2}{2mr^2} + \frac{\gamma}{r}}}$$

$$u = \frac{l}{\sqrt{2m}\, r}$$

$$du = -\frac{l}{\sqrt{2m}}\frac{1}{r^2}\, dr$$

Tir gibanja.

$$\varphi - \varphi_0 = -\int_{u_0}^{u} \frac{du}{\sqrt{E_0 - u^2 + \frac{\gamma\sqrt{2m}}{l} u}}$$

$$E_0 - u^2 + \frac{\gamma\sqrt{2m}}{l} u = E_0 - \left(\frac{\gamma\sqrt{2m}}{2l} - u\right)^2 + \frac{\gamma^2 2m}{4l^2} =$$

$$= \underbrace{E_0 + \frac{\gamma^2 m}{2l^2}} - \left(u - \frac{\gamma\sqrt{2m}}{2l}\right)^2 =$$

$$= \left(E_0 + \frac{\gamma^2 m}{2l^2}\right)\left(1 - \left(\frac{u - \frac{\gamma\sqrt{2m}}{2l}}{\sqrt{E_0 + \frac{\gamma^2 m}{2l^2}}}\right)^2\right)$$

$$\varphi - \varphi_0 = \cancel{\frac{1}{\sqrt{\ }}} \arccos \frac{u - \frac{\gamma\sqrt{2m}}{2l}}{\sqrt{\ }} \Bigg|_{u_0}^{u}$$

$$u_0 = \frac{l}{\sqrt{2m}\, r_0} = \frac{l\, m\gamma}{\sqrt{2m}\, l^2} = \frac{m\gamma}{\sqrt{2m}\, l} = \frac{\gamma\sqrt{2m}}{2l}$$

$$\varphi - \varphi_0 = \arccos \frac{u - \frac{\gamma\sqrt{2m}}{2l}}{\sqrt{\ }} - \arccos 0$$

$$\frac{\pi}{2}$$

$$\sqrt{\ }\, \cos\left(\varphi - \varphi_0 + \frac{\pi}{2}\right) = u - \frac{\gamma\sqrt{2m}}{2l} \qquad u = \frac{l}{\sqrt{2m}}\frac{1}{r}$$

$$u = \frac{\gamma\sqrt{2m}}{2l} + \sqrt{\ }\,\cos\left(\varphi - \varphi_0 + \frac{\pi}{2}\right) \Rightarrow r = \frac{l}{\sqrt{2m}}\,\frac{1}{\frac{\gamma\sqrt{2m}}{2l} + \sqrt{\ }\,\cos\left(\varphi - \varphi_0 + \frac{\pi}{2}\right)}$$

$$r = \frac{p}{1+\varepsilon\cos\varphi} = \frac{\frac{l}{\sqrt{2m}}\cdot\frac{2l}{\gamma\sqrt{2m}}}{1 + \frac{2l}{\gamma\sqrt{2m}}\sqrt{\ }\,\cos\varphi} = \frac{\frac{l^2}{m\gamma}}{1+\varepsilon\cos\varphi}$$

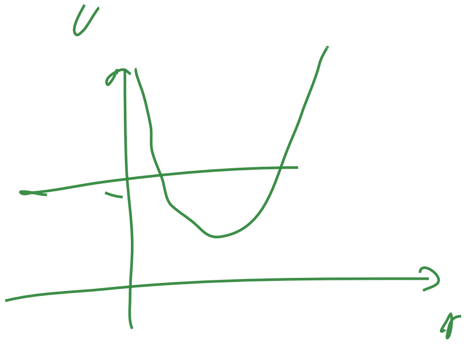
$$\varepsilon = \sqrt{\frac{2l^2}{\gamma^2 m}\left(E_0 + \frac{\gamma^2 m}{2l^2}\right)} = \sqrt{1 + \frac{2l^2}{m\gamma^2}E_0}$$

$$p = \frac{l^2}{m\gamma} \quad ; \quad \varepsilon = \sqrt{\ } \quad < 1$$

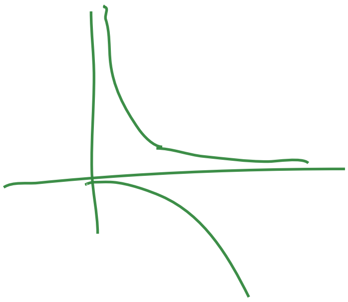
$$r = \frac{p}{1+\epsilon\cos\underbrace{(\theta-\theta_0+\pi/2)}_{\varphi}}, \qquad p = \frac{l^2}{m\gamma}, \qquad \epsilon = \sqrt{1+\frac{2El^2}{m\gamma^2}}$$

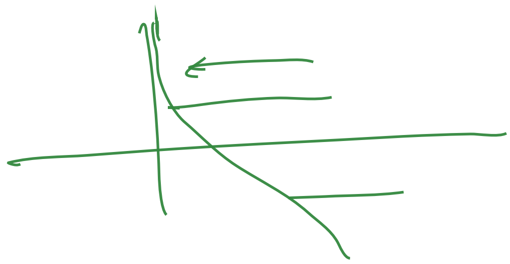
Gibanje v polju potenciala $V = \alpha r^p$

$$V = \frac{l^2}{2mr^2} + \alpha r^p$$

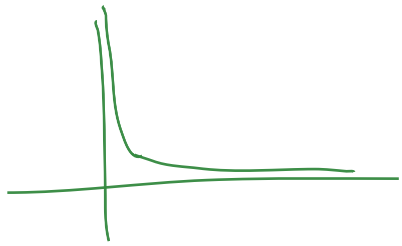


$\alpha < 0$

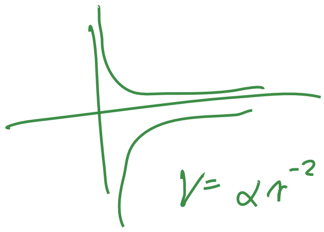




$p = -1$, $\alpha > 0$

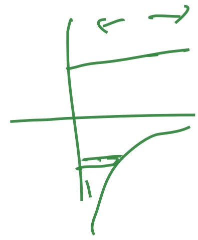


$p = -2$, $\alpha < 0$

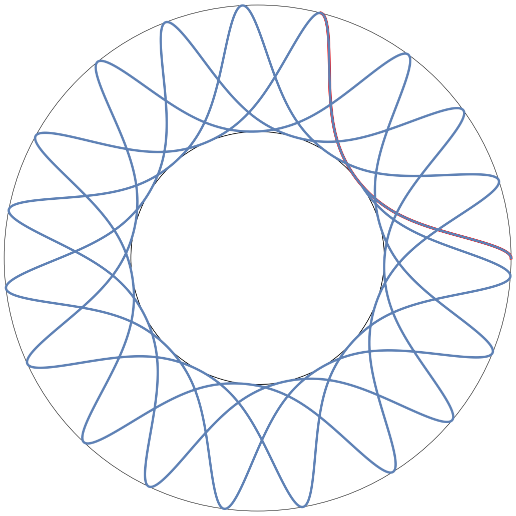


$V = \alpha r^{-2}$

$|\alpha| > \frac{l^2}{2m}$



Kvalitativna analiza gibanja v polju centralne sile



Slika 1: Gibanje v polju centralne sile.

Apsidni radij, pericenter, apocenter, apsidni kot $\Delta\theta$.

Trditev 3. *Tir se dotika apsidnih krožnic.*

Trditev 4. *Tir je simetričen glede na apsidni radij.*

Pogoj zaprtosti tira $W = \Delta\theta/\pi \in \mathbb{Q}$.

Č omejeni tir ni zaprt je gosta množica v kolobarju med apsidnima radijema.

Bertrandov izrek: edina potenciala za katera je vsak omejen tir v okolici krožnega tira zaprt sta gravitacijski in Hookov.