What is a plasma?

* a plasma is a quasi-neutral gas consisting of positively and negatively charged particles which are subject to electric, magnetic and other forces, and which exhibit collective behaviour

* ions and electrons may interact via short range atomic forces during collisions and via long range electro-magnetic forces due to currents and charges in the plasma

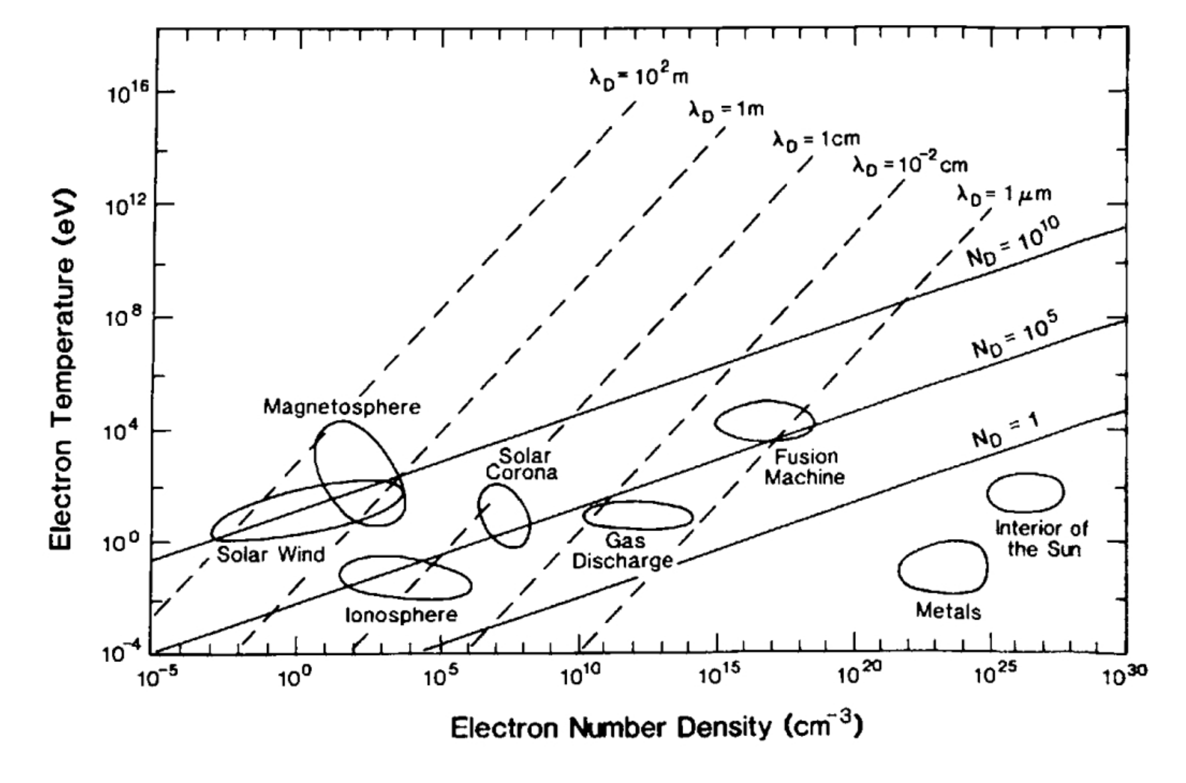
* the long range nature of the electromagnetic forces means that plasma can show collective behaviour, e.g., oscillations, instabilities, …

* plasmas can also contain some neutral particles, which interact with charged particles via collisions or ionisation
Earth’s ionosphere, upper atmosphere, interstellar medium, molecular clouds

* the simplest plasma is formed by ionisation of atomic hydrogen

Plasmas in the Universe

* plasmas form wherever temperatures are high enough or radiation is strong enough to ionize atoms



Plasmas in the Universe

* Earth's and other planets ionosphere and magnetosphere

* Sun's and other stars' armospheres and winds

* comets

* cosmic rays

* interstellar medium

* jets in active galaxies: radio jets and emission

* pulsars and their magnetisphere

* accretion disks around stars

Different ways of describing plasmas

* exact – to specify the state of a plasma by giving the positions and velocities of all the particles and all the fields in all the points in space. For a system of N particles the particle description would be pa phase space of 6N dimensions. In a 1km cube in the interplanetary medium $N \sim 10^{15}$.

* distribution function – $f(\mathbf{x},\mathbf{v})$ in a 6 dimensional phase space. The number of particles in an elemental volume at position (x,v) in phase space is given by: $f(\mathbf{x},\mathbf{v})\, d^3x\, d^3v$. The theory of the evolution of the distribution function is known as plasma kinetic theory.

* MHD – to treat the plasma as a conducting fluid and adapt the equations of fluid dynamics to include the effects of electric and magnetic forces. The resulting theory in its simplest one-fluid form is called MagnetoHydroDynamics. MHD can be derived from the kinetic theory in certain limits, but this involves averaging out almost all the interesting kinetic effects.

Maxwell’s equations

* most astrophysical plasmas are of low density and quantum-mechanical effects can be neglected

* plasmas are dominated by long range forces

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_q}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

Conservation of charge

* in the absence of ionization or recombination one has:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho_q}{\partial t} = 0$$

Lorentz force

* force on a single particle of charge *q* and velocity **v**:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

* for an extended medium there is a force density of $\rho_q \mathbf{E} + \mathbf{j} \times \mathbf{B}$

Magnetohydrodynamics (MHD)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = -\nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \left(p \rho^{-\gamma} \right) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{j} / \sigma$$

* σ is the electrical conductivity

Kinetic theory

* the evolution of the distribution function $f_\alpha(\mathbf{x},\mathbf{v},t)$ for particles of species α in space and time in the absence of collisions is called the Vlasov equation:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{f}_\alpha + \frac{\mathbf{q}_\alpha}{\mathbf{m}_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial \mathbf{f}_\alpha}{\partial \mathbf{v}} = 0$$

* coupled with Maxwell's equations in which the charge and current densities are determined from the particle distributions as

$$\rho_q = \sum_\alpha \iiint_V q_\alpha f d^3 v$$

$$\mathbf{j} = \sum_\alpha \iiint_V q_\alpha \mathbf{v} f d^3 v$$

Frame transformations

* **E** and **B** are frame dependent quantities, in order to transform between frames the appropriate relativistic Lorentz transformation has to be used

* in a 'laboratory' coordinate frame the fields are **E** and **B**

* in another frame which is moving relative to the first at a velocity **u** the fields are **E'** and **B'**

* the fields in the moving frame are given by:

$$\mathbf{E}'_{\|} = \mathbf{E}_{\|}$$
$$\mathbf{B}'_{\|} = \mathbf{B}_{\|}$$

$$\mathbf{E}'_{\perp} = \frac{\mathbf{E}_{\perp} + \mathbf{u} \times \mathbf{B}}{\sqrt{1-(u^2/c^2)}}$$

$$\mathbf{B}'_{\perp} = \frac{\mathbf{B}_{\perp} - \dfrac{\mathbf{u} \times \mathbf{E}}{c^2}}{\sqrt{1-(u^2/c^2)}}$$

Plasma oscillations

* a plasma with equal number of positively and negatively charged particles

* at sufficiently large distances this produces a vanishingly small electic field, since all contributions from positive and negative charges cancel

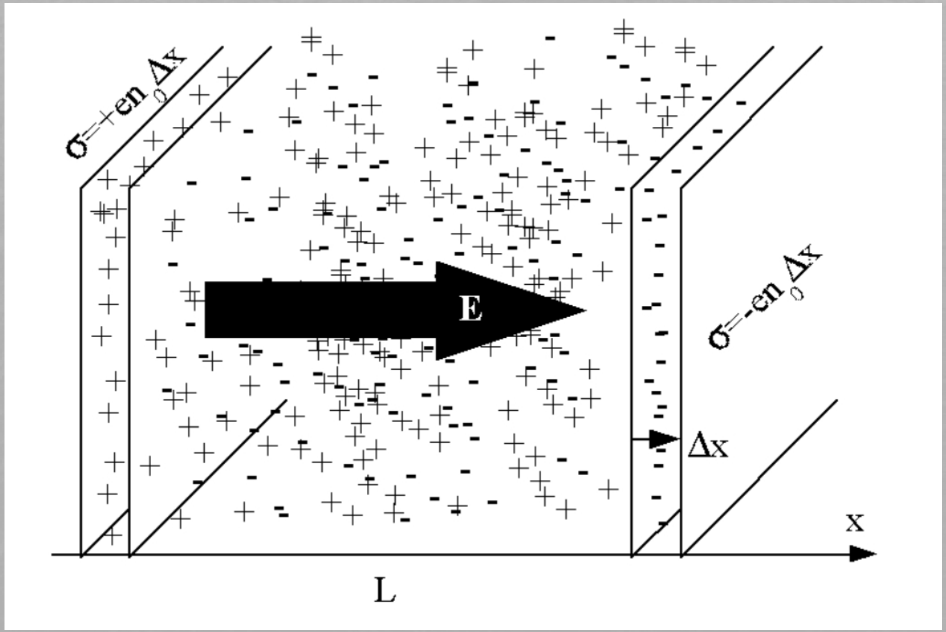
* any departure from charge neutrality leads to a restoring force back towards charge neutrality

* the restoring force leads to a natural oscillation of the plasma, which are called plasma oscillations and occur at a frequency called the plasma frequency

* in a plasma the electrons move much more rapidly than the ions, so they are mostly responsible for this oscillation

Plasma oscillations

* the ions are fixed, and the electrons have no thermal motion
* a planar slab, having thickness L, of electrons is displaced a distance Δx in the x direction, $\Delta x << L$
* if the particle number density is uniform $n_e = n_i = n_0$, then this displacement produces two charge sheets of opposite sign with charge per unit area of $\sigma = en_0\Delta x$
* this is like a parallel plate capacitor, with a resulting electric field $E = \sigma/\varepsilon_0 = en_0\Delta x/\varepsilon_0$



Plasma oscillations

* the electron slab has mass per unit area of $m_e n_0 L$, charge per unit area of $-en_0 L$ and is subject to the electric field E

* equation of motion for the displacement Δx of the electron sheet is:

$$(-e n_0 L)E = m_e n_0 L \frac{d^2 \Delta x}{dt^2}$$

or

$$\frac{d^2 \Delta x}{dt^2} = -\omega_{pe}^2 \Delta x$$

* electron plasma oscillations

Electron Plasma Frequency	$\omega_{pe}^2 = \frac{n_0 e^2}{m_e \varepsilon_0}$

Plasma oscillations

* plasma oscillations are independent of L, they are a direct result of the plasma trying to maintain charge neutrality

* $f_{pe}(kHz) \equiv \omega_{pe}/2\pi \approx 9\sqrt{n_e(cm^{-3})}$

* plasma oscillations can be driven by the natural thermal motions of the electrons

* in this case one can equate the work done by moving the electron sheet through a distance Δx by integrating $\int_0^{\Delta x} e\,E(x)\,dx = \int_0^{\Delta x} \left(\frac{e^2 n_0 x}{\epsilon_0}\right) dx$ with the average energy in thermal agitation in the x-direction: $\frac{e^2 n_0 x^2}{2\epsilon_0} \approx \frac{1}{2} k_B T$

* maximum distance the electron sheet can move due to thermal agitation is: $\Delta x_{max} = \lambda_D$

Debye Length	$\lambda_D = \sqrt{\frac{\varepsilon_0 k_b T}{e^2 n_0}}$

Plasma oscillations

* the Debye length is the spatial scale over which charge neutrality is violated by spontaneous fluctuations

* in order for a charged gas to show collective behaviour the Debye length must be much smaller than the spatial scale of the system

* the number of particles, N_D, within the Debye "sphere" must be large

Debye Number	$N_D = n_0 \lambda_D^3$

* in interplanetary space λ_D is of the order of a few metres and a few *cm* in the solar corona, the number of particles in a Debye sphere, $N_D \sim 10^8$ in each case

Charge shielding

* how a neutral plasma responds to a test charge, and calculating the electric potential φ in the plasma as a result of this additional charge

* in vacuum the potential of the charge would just fall off as 1/*r*

* in a plasma the charge balancing forces result in a redistribution of the plasma charges, so that the potential has the form:

$$\varphi = \frac{A}{r} e^{-r/\lambda_D}$$

* the effect of the test charge, i.e., the electrostatic potential it induces, falls off faster than it would in vacuum – charge shielding

* one sees that the test charge is effectively screened out beyond $r \sim \lambda_D$

* for charge shielding to be effective there must be sufficient particles in a Debye sphere

Collisions, mean free path and collisionless plasmas

* neutral particles have quite small collision cross sections, i.e. effective size as seen by another particle, because they only interact via short range forces, where electron shells have to overlap

* for charged particles the Coulomb force is long range, so collisions in a plasma are in reality a lot of small angle deflections, rather than a few large angle changes

* in an ionized hydrogen plasma electrical and thermal conductivity is effectively controlled by electrons and their interaction frequency with ions

* suppose a plasma has temperature T, and the ions are effectively at rest with respect to the electrons

* the thermal energy of an electron is $3k_bT/2$

* an electron will be affected by a neighbouring ion if the Coulomb potential is of the order, or more than, the electron thermal energy

Collisions, mean free path and collisionless plasmas

* a Coulomb interaction distance, r_C: $$\frac{q^2}{4\pi\varepsilon_0 r_C} \approx \frac{3}{2} k_b T$$

* effective Coulomb cross-section: $$\sigma_C = \pi r_C^2 \approx \pi \left(\frac{q^2}{6\pi k_b \varepsilon_0} \right)^2 \frac{1}{T^2}$$

* in solar wind plasma T = 10^6 K this gives $\sigma_C \sim 10^{-22}$ m^2 which is much larger than the nuclear cross-section of 10^{-30} m^2

* a particle A moving at speed v in a gas of particles B which are assumed at rest, with number density *n*, and σ is the collision cross-section for A on B

* after a collision A is sent off in a random direction

Collisions, mean free path and collisionless plasmas

* the collision frequency is given by σvn, which is a number of particles in a cylinder of cross sectional area σ and length vt divided by the time t taken to traverse the cylinder
* the mean time between collisions is 1/(σvn)
* the mean distance travelled by A between collisions, referred to as the mean free path λ_{mfp} is:

$$\lambda_{mfp} = \frac{1}{\sigma n}$$

* it is possible that λ_{mfp} becomes greater than the size of the system and in this case the plasma is collisionless

* example: in solar wind particles experience only about one collision on their journey from the Sun to the radius of the Earth's orbit

* motion of particles in collisionless plasmas is governed by the behaviour of the particles in the large scale fields

One-fluid MHD equations

* the MHD equations describe the plasma as a conducting fluid with conductivity σ which experiences electric and magnetic forces
* the fluid is specified by a mass density ρ, a flow velocity **V**, and a pressure p, which are all functions of space and time
* in electron-proton plasma the mass density ρ is related to the number density n by $\rho = n\,(m_i + m_e)$
* the MHD equations represent the conservation of mass, the conservation of momentum, and some equation of state i.e., an energy relationship:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\
\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} &= -\nabla p + \mathbf{j} \times \mathbf{B} \\
\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (p \rho^{-\gamma}) &= 0
\end{aligned}
$$

* convective derivative measures the rate of change as seen by parcel of the fluid:

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right)$$

* it sees any instrinsic time variation and additionally variations which arise because it is “convecting” to other regions where, due to spatial gradients, the fluid parameters are different

One-fluid MHD equations

* assumptions: – the energy equation is the adiabatic equation of state with γ being the ratio of specific heats, $\gamma = 5/3$ for a monatomic gas;
– Ohmic heating $\mathbf{j} \cdot \mathbf{E}$ has been ignored, i.e., assumed to be less important than the plasma's thermal energy;
– radiative or other heat losses are also neglected or modelled in an ad hoc fashion by playing with the value of γ;
– the pressure is isotropic which is not generally true when the magnetic field can impose different kinds of motion on the particles parallel and perpendicular to the field direction;
– a possible heat flux has been ignored;
– there is no relative flow between the different species in the plasma, so that the one fluid description is valid;
– while the conductivity is usually large due to the mobile electrons, the absence of collisions has the opposite impact on other transport processes such as viscosity, which we have also therefore neglected

One-fluid MHD equations

* in addition to the equation for the fluid, one also has Maxwell's equations for the fields:

$$\begin{aligned}\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\times\mathbf{B} &= \mu_0\mathbf{j}\\ \nabla\cdot\mathbf{B} &= 0\end{aligned}$$

$$\mu_0\varepsilon_0 = 1/c^2$$

* the displacement current term in the $\nabla \times \mathbf{B}$ equation has been dropped because we are interested in low frequency behaviour
* the equation $\nabla \cdot \mathbf{E}$ has been dropped since we are treating the plasma as a single, charge neutral fluid

* Ohm's law relates the electric field to the current density:

One Fluid MHD Ohm's Law $\quad \mathbf{E} = -\mathbf{V}\times\mathbf{B} + \frac{\mathbf{j}}{\sigma}$

* in the frame moving with the fluid the fluid behaves like a simple conductor with $\mathbf{E}' \propto \mathbf{j}$
* term $-\mathbf{V} \times \mathbf{B}$ is called the motional electric field, and is a consequence of the motion of the fluid and the rule for transforming electric field between frames

One-fluid MHD equations: the induction equation

* taking curl of the one fluid MHD Ohm's law gives:

$$\nabla \times \mathbf{E} = -\nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\sigma} \nabla \times \mathbf{j}$$

* substituting $\mathbf{j} = \nabla \times \mathbf{B}/\mu_0$ from Ampere's law and using the induction law yields:

$$-\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla \times (\nabla \times \mathbf{B})$$

* the double curl can be expanded to give:

$$-\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla (\nabla \cdot \mathbf{B}) - \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

MHD Induction Equation
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

* induction equation, together with the fluid mass, momentum and energy equations form a close set of equations for the MHD state variables (ρ, **V**, *p*, **B**)

Ideal MHD

* when the conductivity is high, $\sigma \to \infty$, such that the electric field is motional electric field only $\mathbf{E} = -\mathbf{V} \times \mathbf{B}$ is known as ideal magnetohydrodynamics

Magnetic field behaviour in MHD: dominant convection

MHD Induction Equation $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

* $\nabla \times (\mathbf{V} \times \mathbf{B})$ dominant – convection

* corresponding to the infinite conductivity limit dubbed ideal MHD, the flow and the field are intimately connected

* the field lines convect with the flow; conductors do not allow new fields to penetrate them, and retain any internal fields they possess: flux freezing

* the flow in turn responds to the field via the $\mathbf{j} \times \mathbf{B}$ force

Magnetic field behaviour in MHD: dominant diffusion

MHD Induction Equation $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

* $1/(\mu_0\sigma)\ \nabla^2$**B** dominant – diffusion

* the induction equation takes the form of a diffusion equation

* the field lines diffuse through the plasma down any field gradients, so as to reduce field gradients

* there is no coupling between the magnetic field and fluid flow

Magnetic field behaviour in MHD: the magnetic Reynold's number

MHD Induction Equation $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}$$

* let's consider the relative sizes of the two right hand side terms in the induction equation, by using dimensional arguments
* we replace ∇ by 1/*L* where *L* is the characteristic scale length, and let *V* denote the typical speed

* the ratio of the convection term to the diffusion term can be expressed by the dimensionless number

Magnetic Reynold's Number $$R_m \equiv \frac{VB/L}{B/\mu_0 \sigma L^2} = \mu_0 \sigma V L$$

* if R_m is large then convection dominates, and the magnetic field is frozen into the plasma
* if R_m is small then diffusion dominates

Flux freezing

* 1942 Alfvén showed that the induction equation for ideal MHD is equivalent to the statement that the field is frozen into the fluid or that the matter of the plasma is fastened to the lines of magnetic force

* by finding out about the plasma flow we are allowed to study the evolution of the field, particularly the topology of the field lines

* if we know how the magnetic field lines evolve, then we can deduce the plasma fluid flow

* ideal MHD induction equation: $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})$$

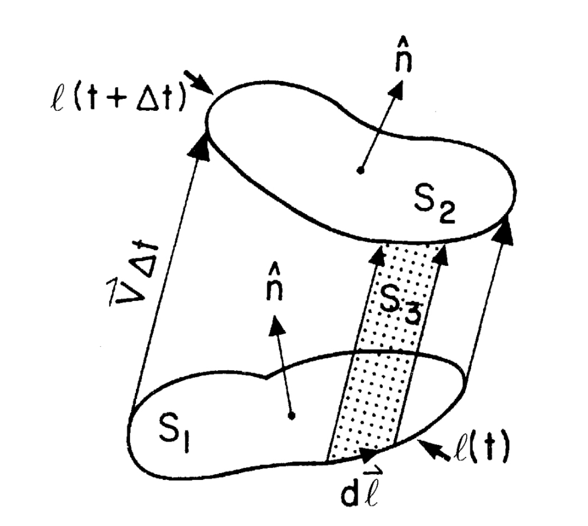
* we want to show that the magnetic flux through a closed loop which moves with the fluid is constant in time

Flux freezing

* the magnetic field through a closed loop *l* is: $\Phi_B \equiv \oint_\ell \mathbf{B} \cdot \hat{n}\, dS$

* the flux freezing law we are trying to establish is then expressed as: $\frac{d\Phi_B}{dt} = 0$

* generalised cylinder formed by the motion of a closed line frozen to the fluid:



Flux freezing

* let Φ_B be the flux enclosed by *l* and Φ_{B1} be the flux through surface S_1, and similarly for S_2 and S_3
* the normal vectors to S_1 and S_2 are chosen to lie on the same side of the surfaces, relative to the fluid flow

$$\frac{d\Phi_B}{dt}=\lim_{\Delta t\to 0}\left(\Phi_{B2}(t+\Delta t)-\Phi_{B1}(t)/\Delta t\right)$$

$$-\Phi_{B1}(t+\Delta t)+\Phi_{B2}(t+\Delta t)+\Phi_{B3}(t+\Delta t)=0$$

* after eliminating Φ_{B1} and at the same time using the definition of flux in expressing Φ_{B2} and Φ_{B3}:

$$\frac{d\Phi_B}{dt}=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\iint_{S_1}(\mathbf{B}(t+\Delta t)-\mathbf{B}(t))\cdot\hat{n}\,dS-\iint_{S_3}\mathbf{B}\cdot\hat{n}\,dS\right]$$

* the first term is:

$$\iint_{S_1}\frac{\partial\mathbf{B}}{\partial t}\cdot\hat{n}\,dS$$

Flux freezing

* we can convert the surface integral over S_3 to one over S_1 by noting that the area element for S_3 can be written $\hat{n}dS$ = **d***l* × **V**Δt:

$$\iint_{S_3} \mathbf{B} \cdot \hat{n}\, dS = \oint_{\ell(t)} \mathbf{B} \cdot (\mathbf{d}\ell \times \mathbf{V}) \Delta t = \oint_{\ell(t)} (\mathbf{V} \times \mathbf{B}) \cdot \mathbf{d}\ell \Delta t$$

* using Stokes theorem to convert the line integral to a surface integral:

$$\iint_{S_3} \mathbf{B} \cdot \hat{n}\, dS = \iint_{S_1} \nabla \times (\mathbf{V} \times \mathbf{B}) \cdot \hat{n}\, dS \Delta t$$

* finally: $$\frac{d\Phi_B}{dt} = \iint_{S_1} \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{B}) \right] \cdot \hat{n}\, dS = 0$$ which establishes the flux freezing theorem

Flux freezing

* the most important implication of the flux freezing theorem is that it imposes constraints on the allowable motion of the plasma

* we can, for example, define a magnetic flux tube by taking a closed loop and moving it parallel to the field it intersects; the surface thus created has zero flux through it, and consequently the fluid elements that form the flux tube at one moment, form the flux tube at all instants

* if two fluid elements are linked by a field line, defined as the intersection of two flux tubes at one instant, then they are always so linked

* if the conductivity σ is not infinite, or equivalently R_m is small, then the flux "thaws" and then slippage of the the field relative to the flow is possible.

Example of flux freezing: field line draping

* a flowing plasma with the field lines initially perpendicular to the flow

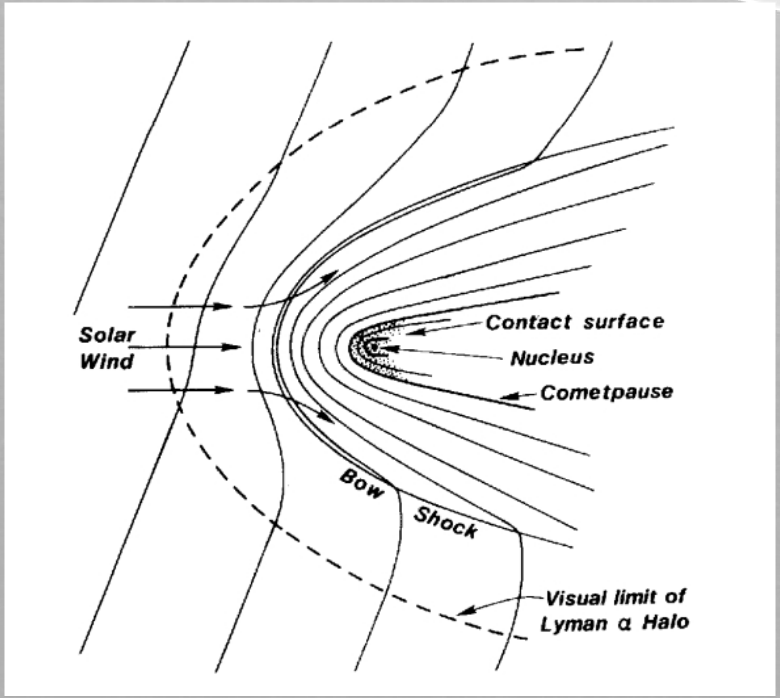
* suppose that there is some region where the flow is slowed, or even stopped, by some kind of obstacle, as on the sketch of the solar wind – comet interaction:

* at the obstacle the field lines are slowed, effectively caught up by the obstacle

* this has the effect of stretching out the field lines behind the obstacle, creating a “tail” in the magnetic field behind the region of slowed flow

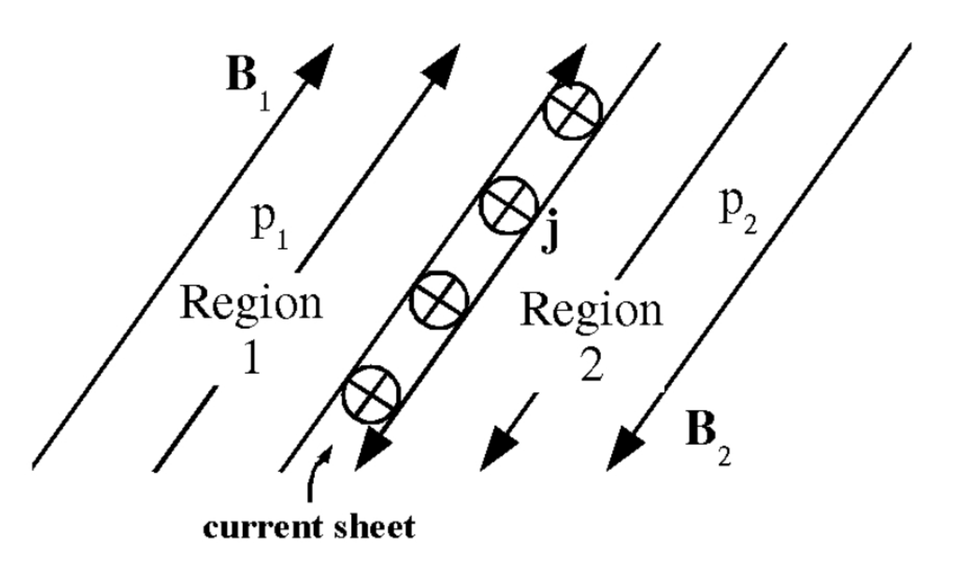
* in the tail the field reverses direction, so there will be a current sheet at the centre of the tail

* Earth, other planets, comets are observed to have Some kind of magnetic tail



The plasma cell model of astrophysical plasmas

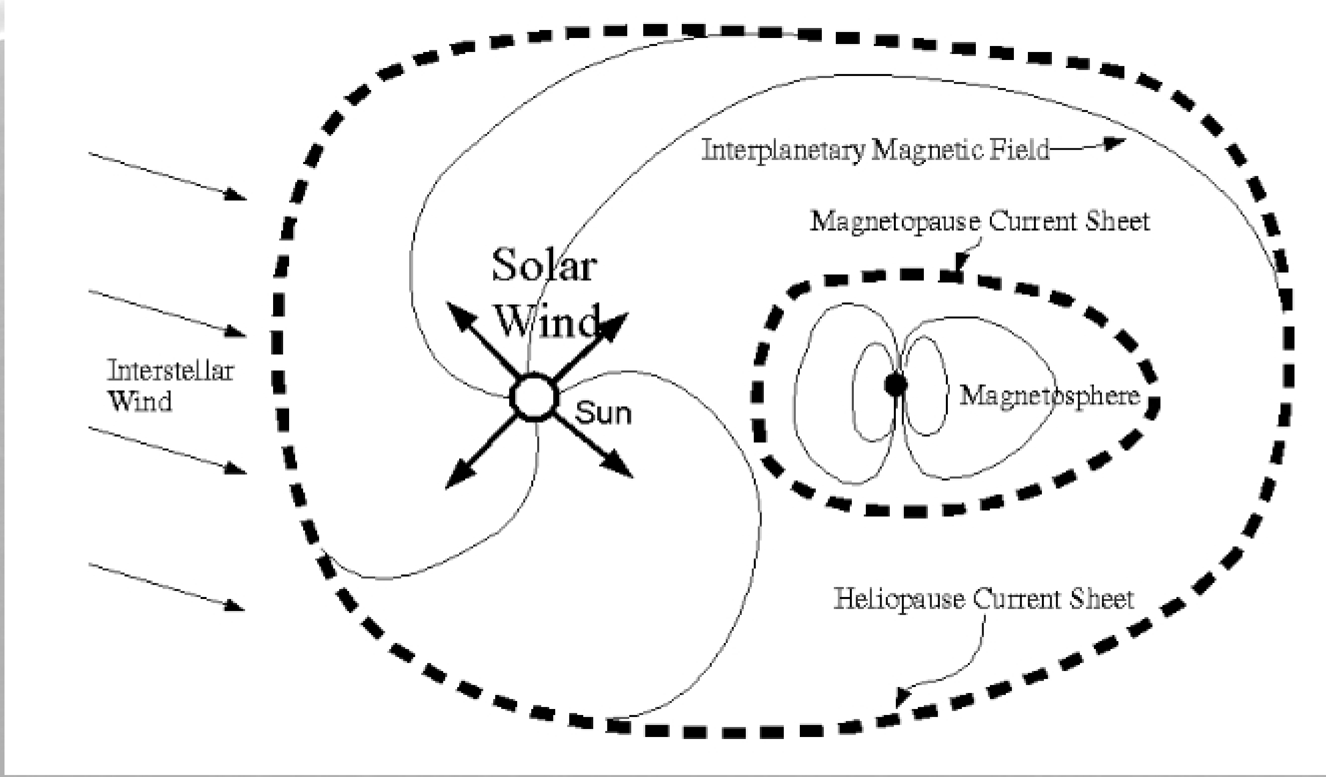
* If R_m is large can we neglect the effects of magnetic diffusion completely?
* large R_m means no cross field plasma mixing: particles are tied to the same field line
* astrophysical plasmas generally come from different sources, such as stellar winds, or clouds of interstellar gas
* a contact between two different plasma regions: in ideal MHD separation of plasmas by a thin boundary layer determined by a pressure balance, since magnetic field changes across it -- current sheet
* astrophysical plasma systems become divided into separate cells containing field and plasma from different sources, separated by current sheets
* in well defined planetary magnetospheres boundary layers have thickness of order 100−1000km and an overall size of the system is of ~100,000km



* diffusion is important at boundary layers and is the mechanism of mass and momentum transfer between the two plasma systems

The plasma cell model of astrophysical plasmas

* illustration of the plasma cell model of the solar system:



Electromagnetic forces in MHD

* the equation of motion for the fluid will generally be of the form:

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right)\mathbf{V} = -\nabla p + (\text{e.m. forces}) + (\text{viscous forces}) + \ldots$$

* treating the plasma as two interpenetrating fluids, one of singly ionised ions and one of electrons, we can write the electromagnetic forces as:

$$\begin{aligned}\mathbf{F}_{e.m.} &= (n_i - n_e)\, e\mathbf{E} + (n_i\mathbf{v}_i - n_e\mathbf{v}_e)\, e \times \mathbf{B} \\ &= \rho_q \mathbf{E} + \mathbf{j}\times\mathbf{B} \\ &= \rho_q \mathbf{E} + \frac{1}{\mu_0}\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right] - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\times\mathbf{B}\end{aligned}$$

* we can now compare the relative magnitude of the three terms on the r.h.s. using dimensional arguments, i.e., substituting $\nabla \Rightarrow 1/L$, $\partial/\partial t \Rightarrow 1/T$, $L/T \Rightarrow V$ and $E \Rightarrow V\,B$

Electromagnetic forces in MHD

* taking the ratio of the middle to last terms: $$\frac{B^2/L\mu_0}{\varepsilon_0 V B^2/T} \sim \frac{T}{\mu_0\varepsilon_0 L V} \sim \frac{c^2}{V^2} \gg 1$$

* the term $\partial\mathbf{E}/\partial t$, the displacement current term, can be neglected provided that the fluid flows are much less than the speed of light

* considering the ratio of the first to middle terms: $$\frac{\varepsilon_0 E^2/L}{B^2/L\mu_0} \sim \frac{V^2}{c^2} \ll 1$$

* the magnetic term dominates over the electrostatic force term for fluid velocities much less than the speed of light

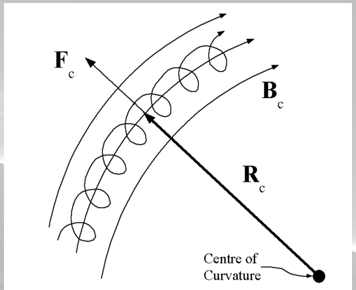
* this justifies using only the magnetic force term in the one fluid MHD equation of motion

* magnetic force can be written:

$$\mathbf{F}_M = \frac{1}{\mu_0}\left[(\nabla\times\mathbf{B})\times\mathbf{B}\right] = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}$$

Electromagnetic forces in MHD

* the last term on the r.h.s. can be written: $$(\mathbf{B}\cdot\nabla)\mathbf{B} = \hat{b}\frac{\partial}{\partial s}\left(\frac{B^2}{2}\right) - B^2\frac{\mathbf{R_c}}{R_c^2}$$



* splitting the gradient: $$\nabla = \nabla_\perp + \hat{b}\frac{\partial}{\partial s}$$

* finally:

MHD Magnetic Forces $$\mathbf{F}_M = -\nabla_\perp\left(\frac{B^2}{2\mu_0}\right) - \frac{B^2}{\mu_0}\frac{\mathbf{R}_c}{R_c^2}$$

* the magnetic force can be resolved into two components: a force perpendicular to the magnetic field which behaves like a pressure, i.e. it is the gradient of a scalar quantity $B^2/2\mu_0$, and a force towards the instantaneous centre of curvature which depends on the curvature and the field magnitude, which is the physical equivalent of a tension force B^2/μ_0 acting along the field lines

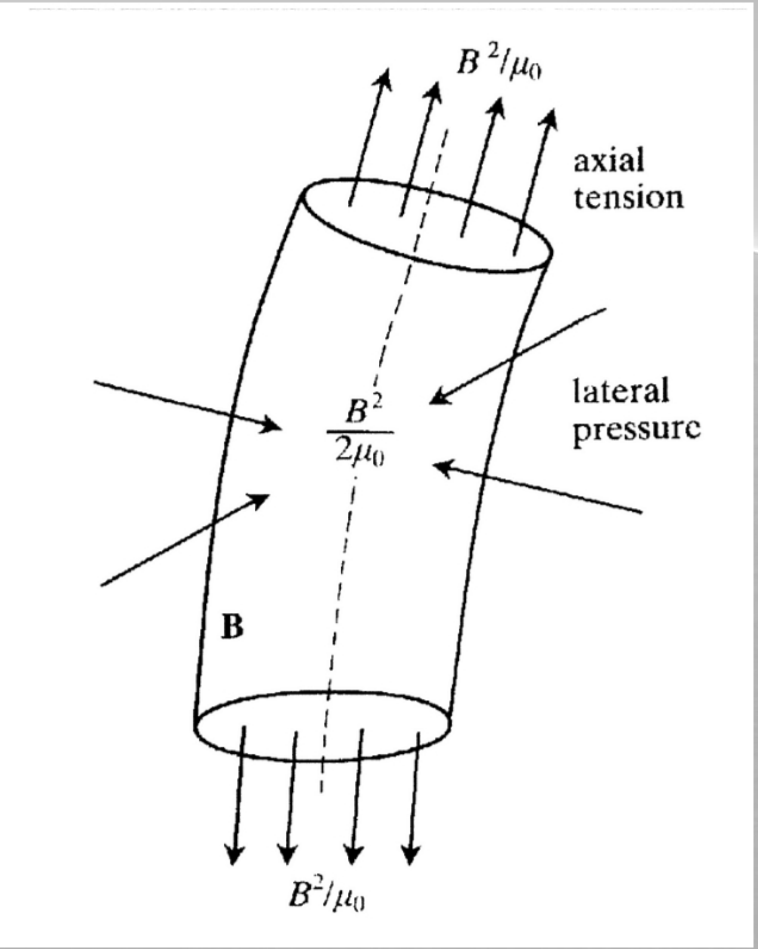
Electromagnetic forces in MHD

* forcing the field lines together results in an opposing perpendicular pressure force

* trying to bend the field lines results in an opposing tension force

* these different kinds of forces lead to different kinds of dynamics

* magnetic pressure and tension forces acting on a magnetic flux tube:



MHD waves

* from the two kinds of restoring force due to the magnetic field one can find a number of wave modes in MHD, investigate the equilibria and instabilities

* from an analysis of the linear wave modes we shall find that, in place of the sound waves associated with a classical gas, there are three distinct MHD waves:

Alfvén waves – the restoring force is due entirely to the tension associated with the field lines; the wave is transverse and essentially magnetic, with no compression of the matter

Fast mode waves – the field magnitude and gas pressure both vary, their variation is in phase, and the waves propagate faster than an Alfvén wave

Slow mode waves – the field magnitude and gas pressure both vary, their variation is out of phase, and the waves propagate slower than an Alfvén wave

MHD waves

* we investigate the MHD equations for small amplitude, plane wave solutions
* we substitute into the MHD equations:

$$
\begin{aligned}
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1 \\
\mathbf{V} &= \mathbf{V}_0 + \mathbf{V}_1 \equiv \mathbf{V}_1 \\
\rho &= \rho_0 + \rho_1 \\
p &= p_0 + p_1
\end{aligned}
$$

* subscript "0" equilibrium values
* subscript "1" represents small, linear perturbations
* higher order perturbations are ignored
* we work in a frame where $\mathbf{V}_0 = \mathbf{0}$
* the equilibrium is assumed to be uniform: ρ_0 = const, $\mathbf{B}_0$ = const and p_0 = const
* we linearize by substituting into the MHD equations, cancel the terms which appear in the equilibrium solutions to obtain:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{V}_1 = 0$$

MHD waves

$$\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} = -\nabla p_1 + \frac{1}{\mu_0}(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{V}_1 \times \mathbf{B}_0)$$

$$p_1 = c_s^2 \rho_1$$

* the sound speed $c_s^2 = \gamma\, p_0/\rho_0$
* we look for plane wave solutions: $Q_1 = \delta Q e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$
* Q_1 represents one of the fluid perturbation quantities defined above, **k** is the wave vector and ω is the angular frequency of the oscillation
* substituting and equating coefficients of the same exponent will simply give the same result as the following substitution in the original equations: $\nabla \Rightarrow i\mathbf{k}$ and $\partial/\partial t \Rightarrow -i\omega$
* the equations for the perturbations become:

$$-\omega\delta\rho + \rho_0 \mathbf{k}\cdot\delta\mathbf{V} = 0$$

$$-\rho_0\omega\delta\mathbf{V} = -\mathbf{k}\delta p + \frac{1}{\mu_0}(\mathbf{k}\cdot\mathbf{B}_0)\,\delta\mathbf{B} - \frac{1}{\mu_0}(\delta\mathbf{B}\cdot\mathbf{B}_0)\,\mathbf{k}$$

$$-\omega\delta\mathbf{B} = (\mathbf{k}\cdot\mathbf{B}_0)\,\delta\mathbf{V} - (\mathbf{k}\cdot\delta\mathbf{V})\,\mathbf{B}_0$$

$$\delta p = c_s^2 \delta\rho$$

MHD waves

* combining the equations for perturbations gives:

$$\delta p = \frac{\rho_0 c_s^2}{\omega} \mathbf{k} \cdot \delta \mathbf{V}$$

* substituting δp and **δB** into 3rd equation gives equation for δ**V**:

$$\begin{aligned} -\rho_0 \omega \delta \mathbf{V} = \quad & -\frac{\rho_0 c_s^2}{\omega} (\mathbf{k} \cdot \delta \mathbf{V}) \mathbf{k} \\ & -\frac{1}{\mu_0} (\mathbf{k} \cdot \mathbf{B}_0) \left[\frac{(\mathbf{k} \cdot \mathbf{B}_0)}{\omega} \delta \mathbf{V} - \frac{(\mathbf{k} \cdot \delta \mathbf{V})}{\omega} \mathbf{B}_0 \right] \\ & +\frac{1}{\mu_0} \left[\frac{(\mathbf{k} \cdot \mathbf{B}_0)}{\omega} (\mathbf{B}_0 \cdot \delta \mathbf{V}) \mathbf{k} - \frac{(\mathbf{k} \cdot \delta \mathbf{V})}{\omega} B_0^2 \mathbf{k} \right] \end{aligned}$$

* this can be written in matrix form, **A** · δ**V** = **0**, where the matrix $\mathbf{A}(\mathbf{k},\omega,\mathbf{B}_0,\rho_0,p_0)$
* we shall here consider in more detail a few special cases related to choices of the orientation of the wave-vector **k** with respect to B_0

MHD waves propagating parellel to B_0

* $\mathbf{B}_0 = B_0\hat{z}$ and **k** are along the *z*-direction

* we consider only the z-component of the previous equation -- both terms in the square brackets vanish, and one is left with: $\omega^2 \delta V_z = k^2 c_s^2 \delta V_z$

* for $\delta V_z \neq 0$ this implies $\omega^2 = k^2 c_s^2$

* one finds that $\delta V_x = 0 = \delta V_y$

* $\boldsymbol{\delta V} \parallel \mathbf{k}$, i.e. the waves are longitudinal, with a phase speed:

Sound Wave Dispersion Relation	$\omega/k = \pm c_s$

MHD waves propagating parellel to B_0

* now we consider only the transverse component of the same equation:

$$-\rho_0\omega\delta\mathbf{V}_\perp = \mathbf{0} - \frac{kB_0}{\mu_0}\left[\frac{kB_0}{\omega}\delta\mathbf{V}_\perp\right] + \mathbf{0}$$

* for $\delta V_\perp \neq 0$ we have:

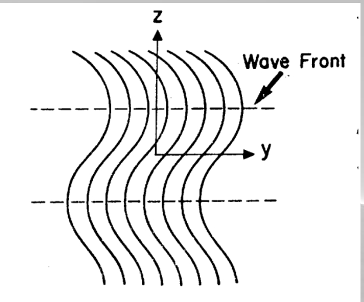
Alfvén Wave Dispersion Relation $\omega^2 = k^2 v_A^2$

where $v_A^2 = B_0^2 / (\mu_0\, \rho_0)$ is the square of the Alfvén speed

* the perturbation $\delta\mathbf{V}$ is perpendicular to **k** and $\mathbf{B}_0$, transverse or shear wave

* the phase speed $\omega / k = \pm v_A$

* for the Alfvén wave: $\delta\mathbf{B}_\perp = -\frac{kB_0}{\omega}\delta\mathbf{V}_\perp$



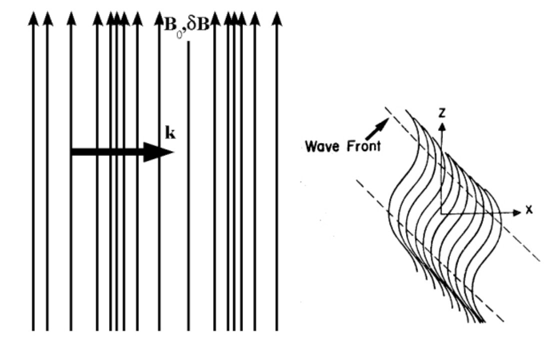
Alfvén wave propagating parallel to the background magnetic field
*the waves are like waves on a string, due to the field line tension
*$\delta\rho = 0 = \delta p$, so there is no compression of the plasma
* Alfvén wave is thus essentially magnetic

MHD waves perpendicular to B_0

* **k** ⊥ $\mathbf{B}_0$ gives $\delta V_z = 0$ and the perpendicular component of the equation gives:

$$-\rho_0 \omega \delta \mathbf{V}_\perp = -\frac{\rho_0 c_s^2}{\omega}(\mathbf{k}\cdot\delta\mathbf{V}_\perp)\mathbf{k} - \mathbf{0} - \frac{(\mathbf{k}\cdot\delta\mathbf{V}_\perp)}{\mu_0 \omega} B_0^2 \mathbf{k}$$

* $\delta V_\perp$ is parallel to **k**: $\omega^2 \delta V_\perp = k^2 (c_s^2 + v_A^2) \delta V_\perp$



fast magnetosonic mode propagating both perpendicular and obliquely to B_0 and reveals the compression in the field

Fast Magnetosonic Dispersion Relation	$\omega^2 = k^2 (c_s^2 + v_A^2)$

* fast mode magnetosonic waves

* both B and p show compression

* the fast mode propagates with a phase velocity:

$$\omega / k \equiv v_f = \sqrt{c_s^2 + v_A^2}$$

General case: k B$_0$ = $kB_0 \cos\theta$

* three waves because there are three restoring forces: magnetic tension, magnetic pressure and thermal pressure

* dispersion relations for these three modes:

Alfvén Wave	$\omega^2 = k^2 v_A^2 \cos^2\theta \equiv k_z^2 v_A^2$
Fast and Slow Magnetosonic Waves	$\omega^2 = k^2 \frac{c_s^2 + v_A^2}{2}\left[1 \pm \sqrt{1 - \frac{4c_s^2 v_A^2 \cos^2\theta}{(c_s^2 + v_A^2)^2}}\right]$

MHD instabilities

* instabilities arise when for some small perturbation the perturbation grows and the system departs further from the equilibrium state toward one of lower energy

* interchange modes – lines are wrapped around plasma in a concave manner
* Rayleigh-Taylor instability – plasma is supported by a field against gravity, which may create structure in prominences
* sausage and kink modes of a flux tube
* Kelvin-Helmholtz instability - plasma flows over a magnetic surface
* resistive modes of a sheared magnetic field -- drive reconnection
* convective instability – when a temperature gradient is too large, concentrates flux tubes in the photosphere
* radiative instability – creates cool loops and prominences up in the corona
* magnetic buoyancy instability –magnetic field in a stratified medium in which the density decreases with height; buoyancy causes flux tubes to rise through the solar convection zone

Shock waves

* a shock is an abrupt transition between supersonic and subsonic flows

* when plasma flows exceed speed of sound or Alfvén speed

* in front of all planets in the solar system, in the magnetotails, formed in the solar corona and solar wind

* in shocks plasma and fields go through dramatic changes: in density, temperature, field strength, flow speed

* they are involved in a wide range of plasma phenomena

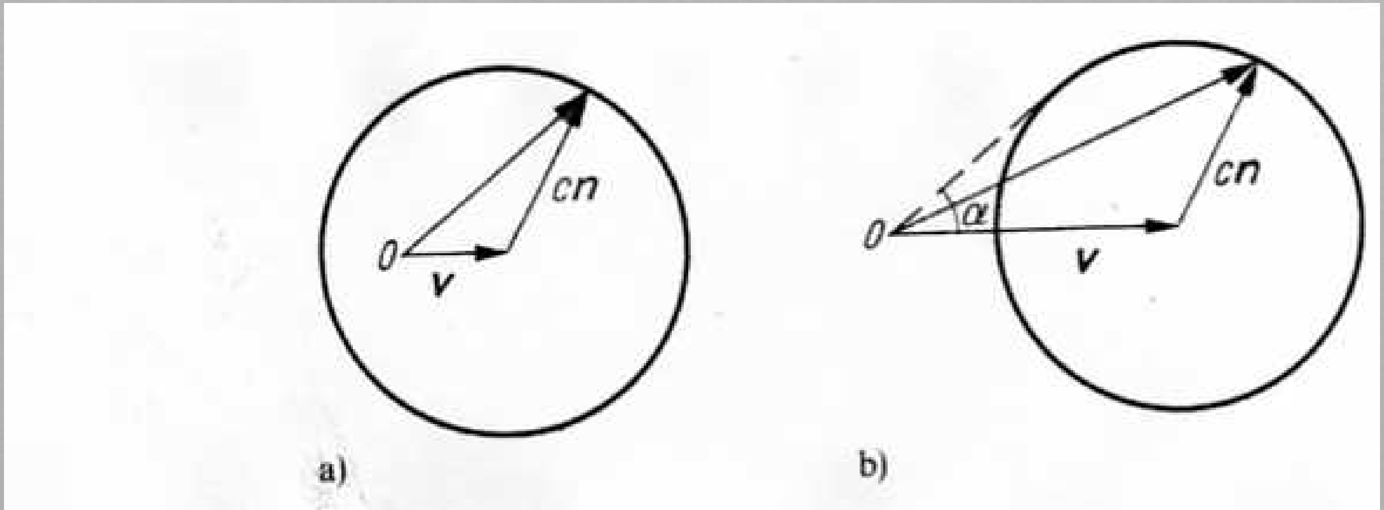
Object in supersonic flow

* if a flow is subsonic, information about the object can transmitted via sound waves against the flow
  flow can respond to the information and is deflected around obstacle in a laminar fashion

* if a flow is supersonic, signals get swept downstream and cannot inform upstream flow about presence of the object
  a shock is launched which stands in upstream flow and effetcs a super- to sub-sonic transition
  the subsonic flow behind the shock is then capable of being deflected around the object

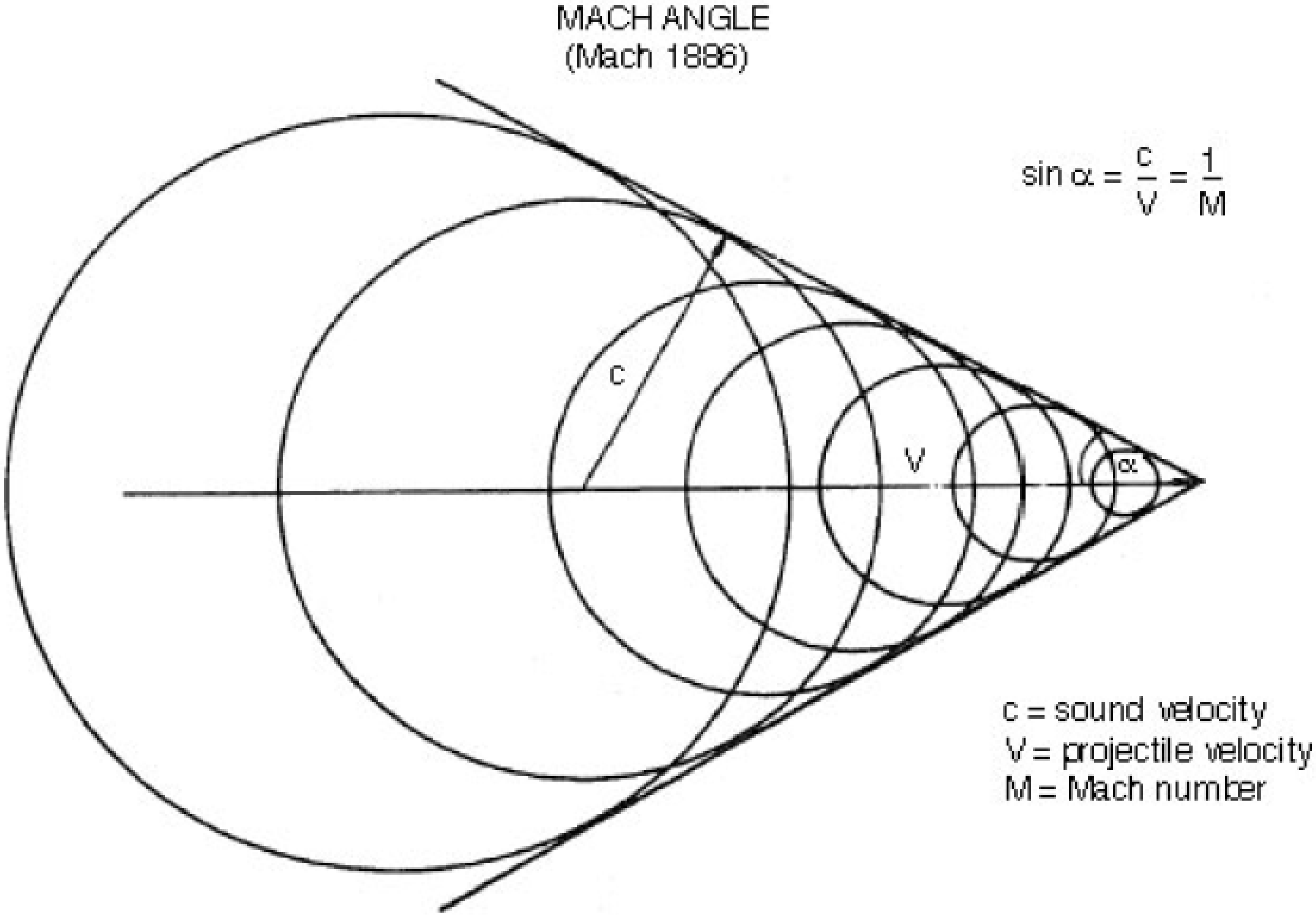
Mach's surface

* fluid moves with velocity **v**; a disturbance occurs at 0 and propagates with the velocity of sound, c, relative to the fluid
* velocity of disturbance relative to 0 is: **v** + c **n,** where **n** is unit vector in any direction
 a) v<c : a disturbance from any point in a subsonic flow eventually reaches any point
 b) v>c: a disturbance from position 0 can reach only the area within a cone given by opening angle 2α, where $\sin\alpha = c / v$
* a surface disturbance can reach is called Mach's surface



Mach's surface

Ernst Mach

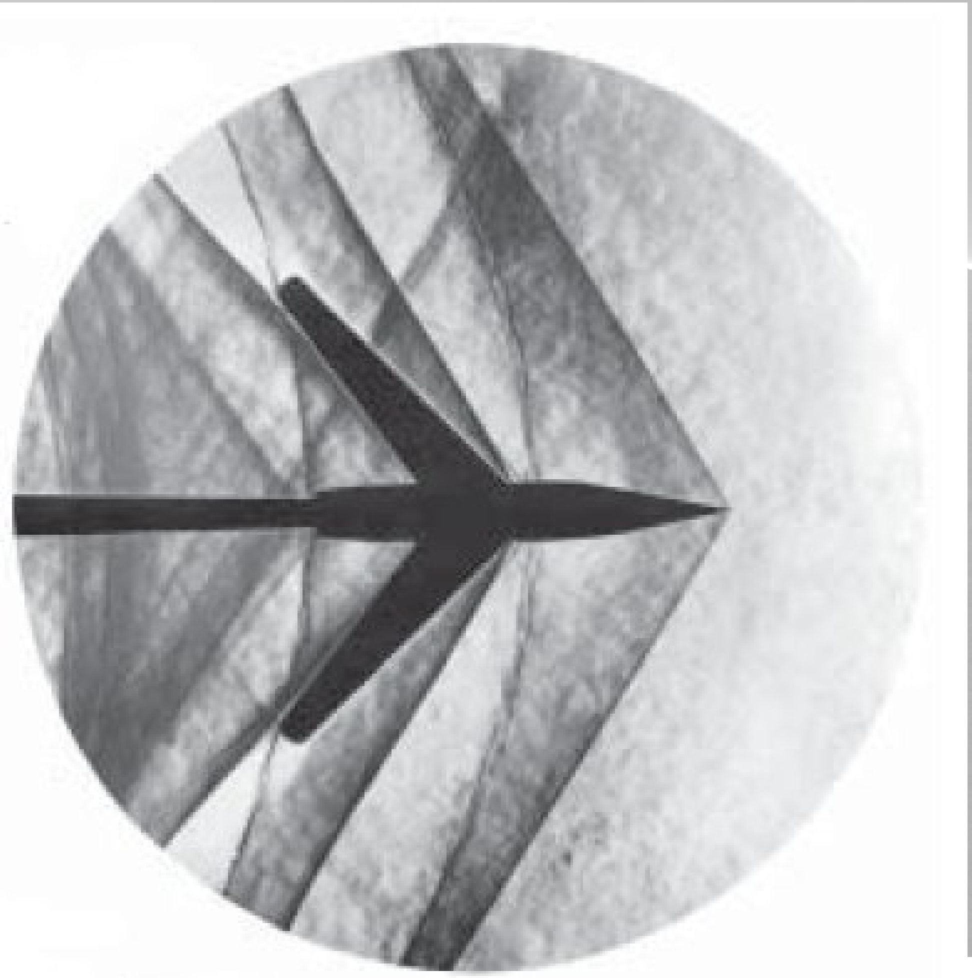


Example of a gas dynamic shock

* when an obstacle such as aircraft travels through air faster than the speed of sound

* when aircraft moves forward the air ahead of it must be diverted around it, so there is a layer of subsonic air flow adjacent to the obstacle pressure force propagates away from the obstacle at the sound speed would be swept downstream and could not affect the flow ahead of the obstacle if this was not the case

* relative to the obstacle the distant air flow is supersonic, so there has to be a transition to the subsonic flow close to the obstacle

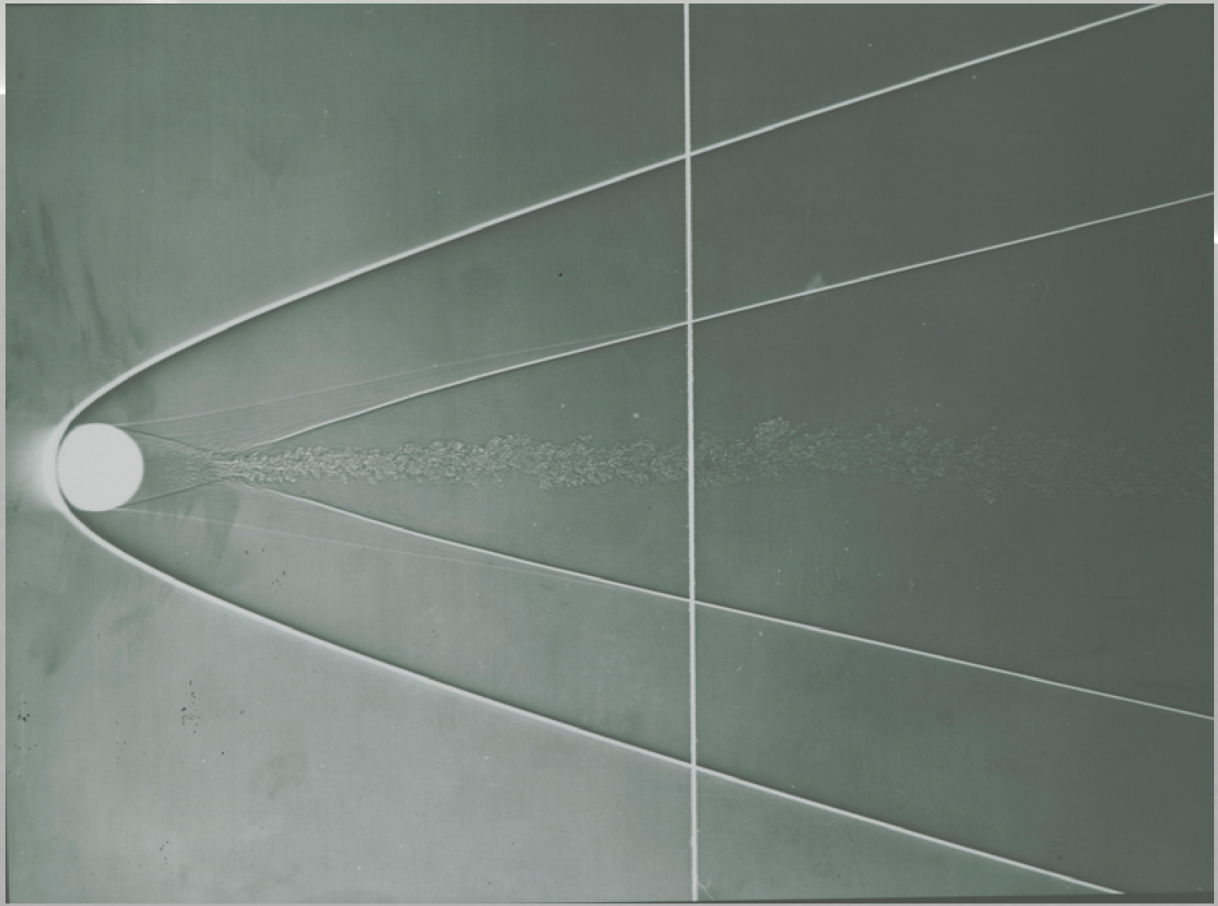


Example of a gas dynamic shock

* a sphere moving through the air at a supersonic speed

* the leading shock wave remains detached from the object

* a second shock wave forms farther back attached to the sphere



Shock waves: historical note

* the study of shocks began with ordinary gas dynamics in the late XIXth century

* 1940s development of high performance aircrafts
  most of the everyday notions of what a shock wave is come from supersonic aircraft or explosive blasts

* 1950s plasma shocks in fusion plasmas and shocks caused by explosions in the upper atmosphere

* realisation that shocks exist in collisionless plasmas such as found in the interplanetary space and in other more exotic astrophysical objects

* early 1960s truly collisionless shocks in the interplanetary space first observed
  confirmation that they are relatively stable structures

* since the start of spaceflight many observations are collected
  detailed observations of the particle distributions and fields

Shock waves

* in ordinary gas collisions serve to transfer momentum and energy among the molecules and provide the coupling which allows the sound wave to exist

* a gas dynamic shock has a width of the order of a mean free path between collisions

* in collisionless plasma the collisional coupling is absent: the mean free path between the collisions is greater than the size of the system

* in the solar wind the collision mean free path is, as calculated from the kinetic theory, about 1AU; however the thickness of the Earth's bow shock is observed to be only 100 – 1000 km

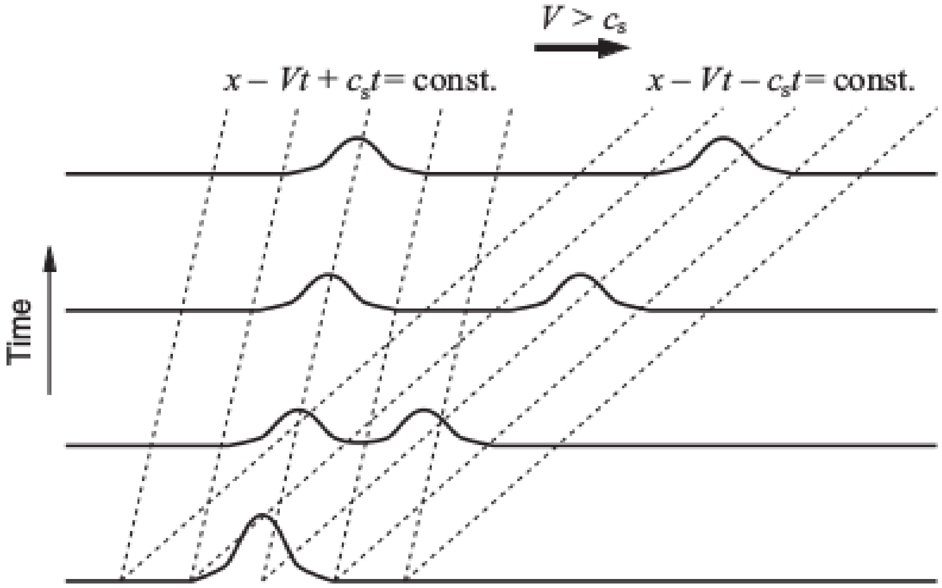
* in a collisionless plasma shock there is no single mechanism which controls the shock dissipation and hence length scales

Shocks in gasses

* a gas which supports sound waves
* small-amplitude linear pressure perturbations described by: $\frac{\partial^2 p}{\partial t^2} - c_s^2 \frac{\partial^2 p}{\partial x^2} = 0,$

* D'Alembert's solution: $p(x,t) = f(x - c_s t) + g(x + c_s t),$

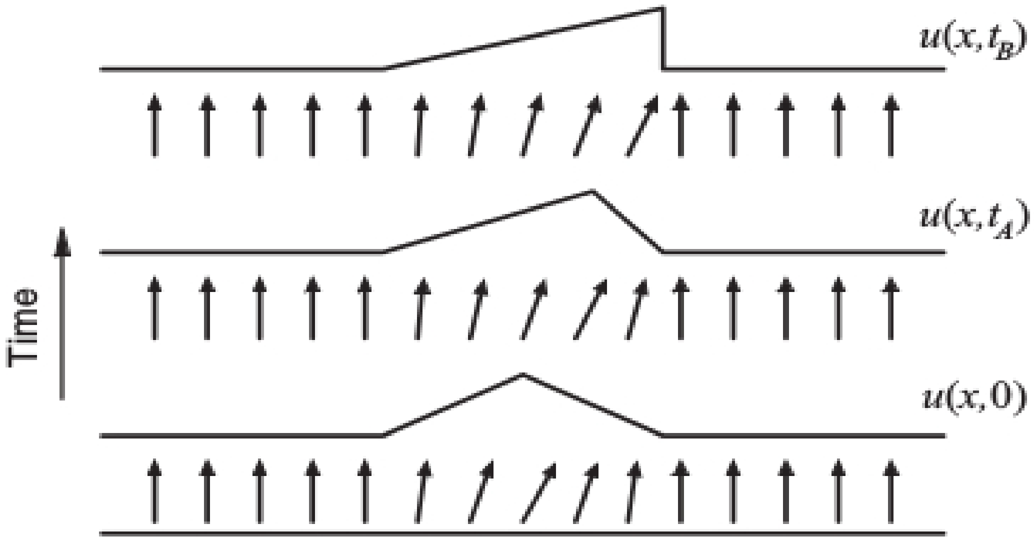
* in supersonic flow both parts of the wave solution are convected downstream



Shocks in gasses

* Burgess' equation with zero viscosity:

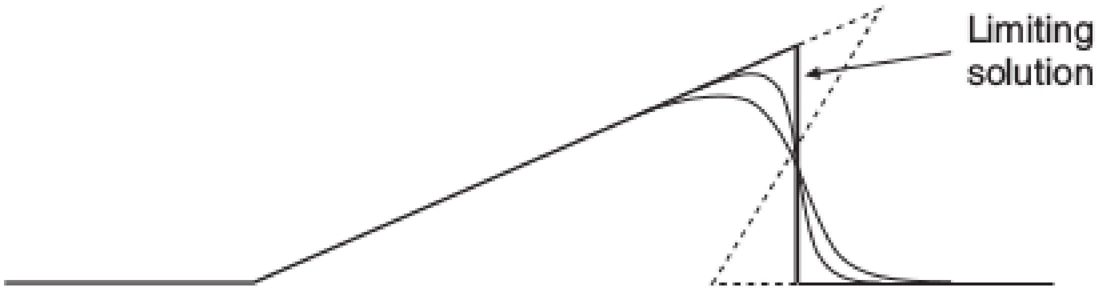
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$



arrows indicate slopes
of the characteristics

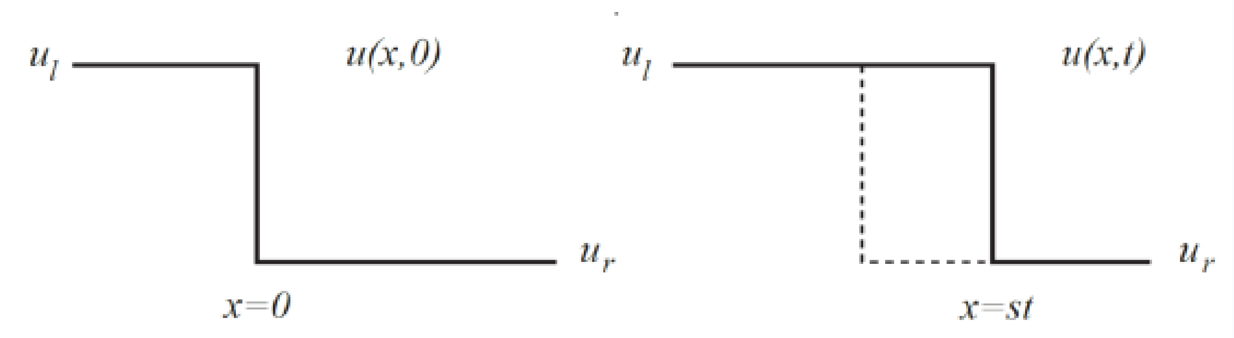
* with dissipation included:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}.$$



Shocks in gasses

* discontinous solutions of Burgess' equation



$$u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases}$$

Shock speed s

$$s = \frac{1}{2}(u_l + u_r)$$

Conservation form:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0$$

Stability requires characteristics sloping *towards* shock

$$u_l > u_r$$

Shocks in gasses

* shock is (near-) discontinous solution linking two thermodynamic states

* Rankine-Hugoniot relations provide unique way of describing downstream state in terms of upstream state and shock parameters in a gas

* a shock is characterized by the upstream flow speed normal to the shock in the shock frame

* the sonic Mach number is: $M_{cs} = \frac{V_{\rm u}}{c_{\rm s}},$

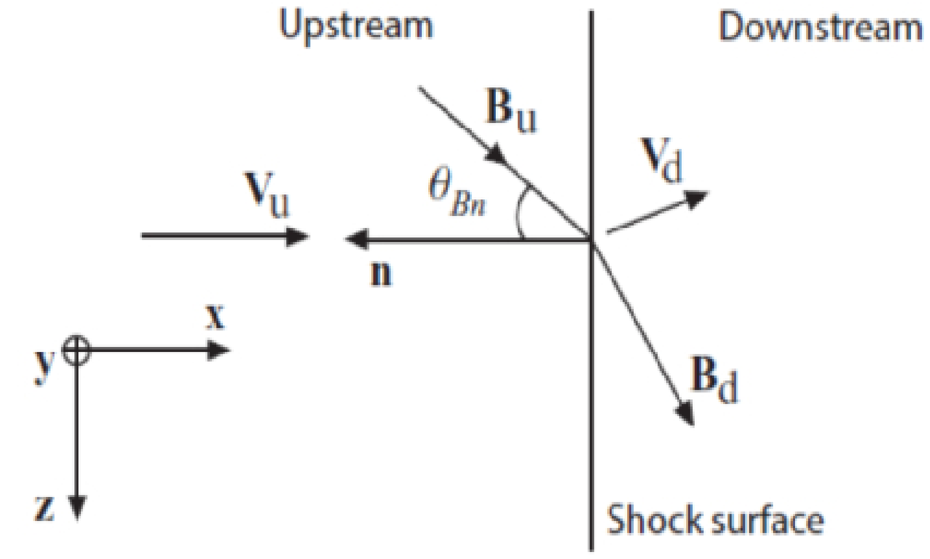
Shocks in MHD plasma

* MHD is a fluid model of a plasma and has many properties of the gas dynamic system

* a difference in MHD is that the wave speeds are anisotropic, consequently Mach number and MHD shocks are characterised by the magnetic field geometry at the shock

* shock normal angle, θ_{Bn}, is found to have a strong influence on the shock jump conditions in MHD and the shock structure in a collisionless plasma

* the Alfven Mach number is: $M_A = \frac{V_u}{v_A}$



Shocks in MHD plasma

* magnetic field geometry at the shock, which is parametrized by the shock normal angle, θ_{Bn}:

perpendicular shock at $\Theta_{bn} = 90°$

parallel shock at $\Theta_{bn} = 0°$

quasi-perpendicular at $\Theta_{bn} > 45°$

quasi-parallel at $\Theta_{bn} < 45°$

* Alfvén Mach number: $M_A = \frac{V_u}{v_A}$,

* fast and slow magnetosonic Mach number: $M_f = \frac{V_u}{v_f}$ $M_s = \frac{V_u}{v_s}$

* fast- and slow-mode wave speeds in the direction normal to the shock surface:

$$\left.\begin{matrix} v_f \\ v_s \end{matrix}\right\} = \left\{\frac{1}{2}\left[v_A^2 + c_s^2 \pm \sqrt{(v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 \cos\theta_{Bn}}\right]\right\}^{1/2}$$

Shocks in collisionless plasmas

* can a shock exist in a plasma where there is no dissipation due to Coulomb collisions?

* does wave steepening lead to discontinuous solutions as in gas dynamics?

Shocks in collisionless plasmas

* plasma can support several different types of wave, involving fields and particles

* most space plasmas are *collisionless*: the media are either so rarefied or hot that Coulomb collisions happen infrequently and do not play an important role

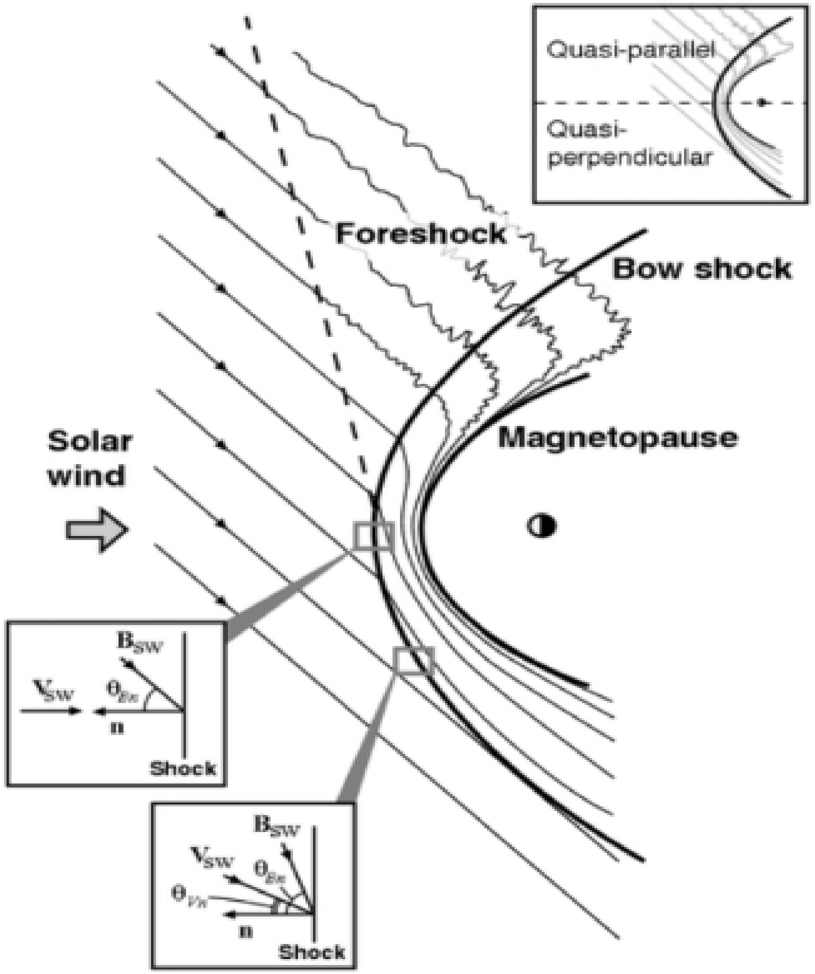
* lack of collisions in a plasma causes that different types of particles, e.g. protons and electrons, can have different temperatures

* particle distribution functions can be different from Maxwellian, so that the concept of temperature has to be broadened to the “kinetic temperature”, i.e. a velocity moment of the particles' distribution function

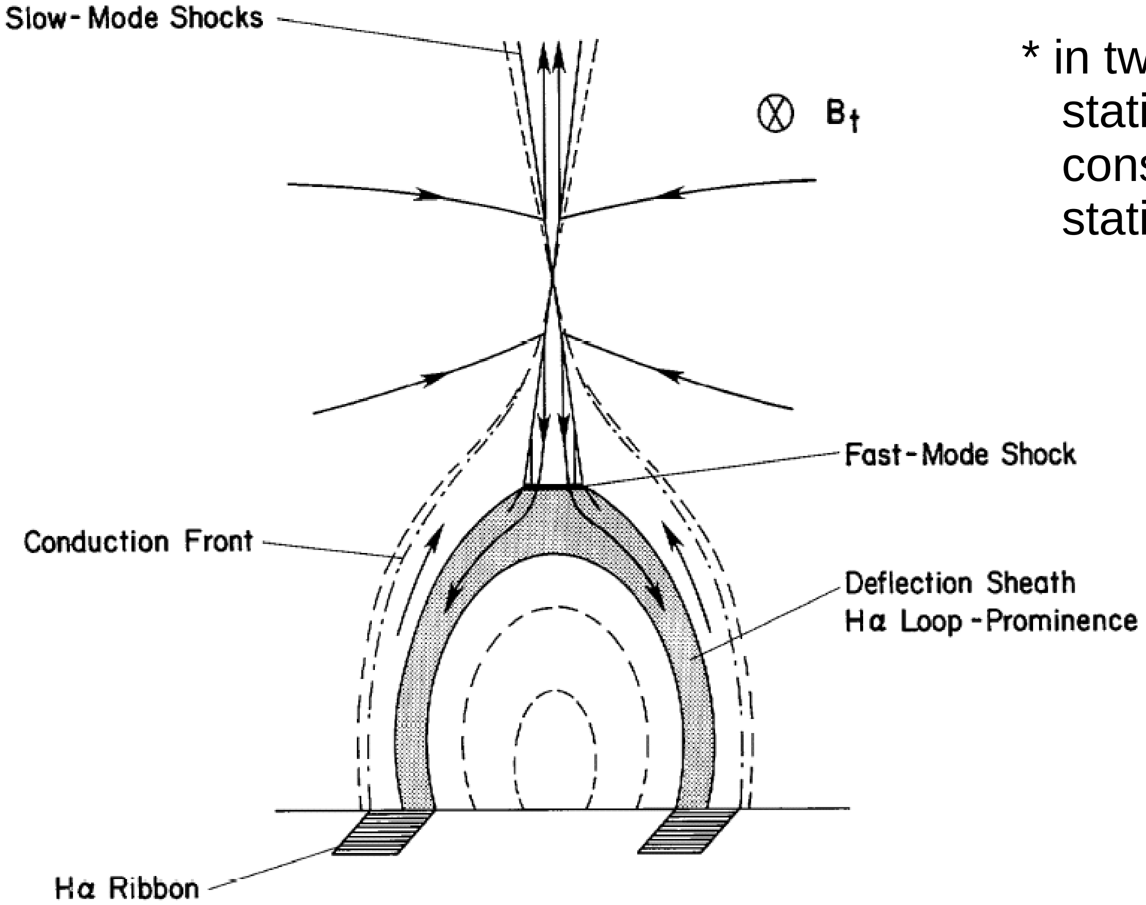
* the important roles of the magnetic and electric fields in a plasma also means that the distribution function may no longer be isotropic in velocity space

Example of collisionless shock: Earth's bow shock

* a standing shock in the solar wind ahead of the Earth's magnetosphere
* the magnetic field of the Earth forms an obstacle to the supersonically flowing solar wind
* the bow shock makes the solar wind transition to subsonic speeds, so that it can flow around the magnetosphere
* it has a curved shape, symmetrical about the Sun-Earth line
* the nose of the bow shock is at ~15R_E
* it is few R_E ahead of the magnetosphere, stand-off distance
* the region of subsonic solar wind behind the bow shock is called the magnetosheath
* typically the IMF is at an angle 45° to the Sun-Earth line
* all of the planets with magnetospheres or ionospheres have bow shocks in front of them

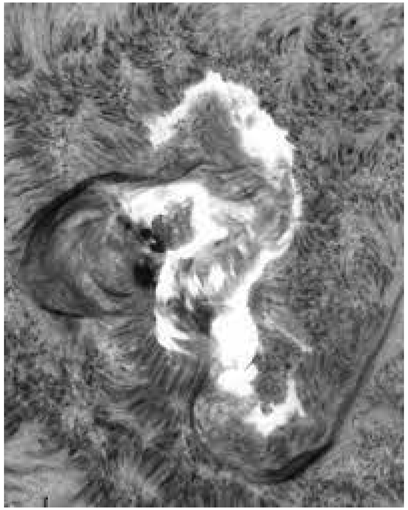


Example of collisionless shock: solar flare



* in two-ribbon flare two pairs of stationary slow mode shocks are considered to play a role and a stationary fast mode shock

ribbons in H_α



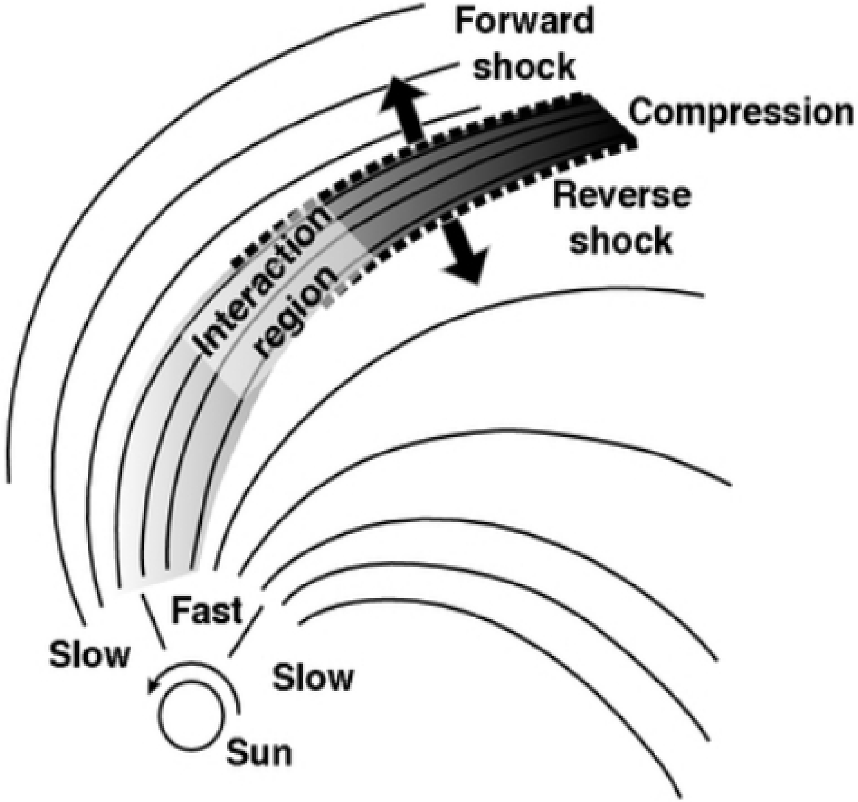
Example of collisionless shock: corotating interaction region

* flow interaction shocks form due to the changes in the flow at the solar wind source

* they propagate with the solar wind itself and are known as interplanetary, IP, shocks

* the Sun produces supersonic solar wind with a speed which depends on type of coronal source region, which is typically in a ‘fast’ or ‘slow’ state

* if at some heliolatitude a source region of fast wind trails behind a source of slow solar wind then solar rotation and the near radial expansion of the solar wind ensures that along radial direction the slow-stream is followed by a fast-stream



* ahead and behind the stream interface a compression forms which can evolve into shocks

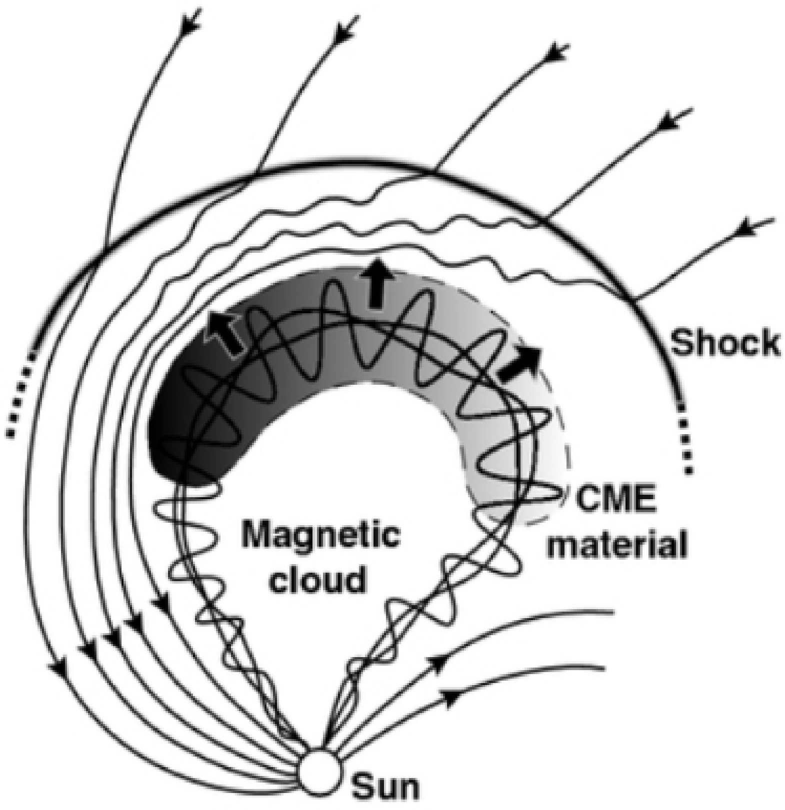
Example of collisionless shock: impulsive IP shock

* interplanetary, IP, shocks can be formed also impulsively when a sudden change of a speed or density occurs at the coronal source of the solar wind

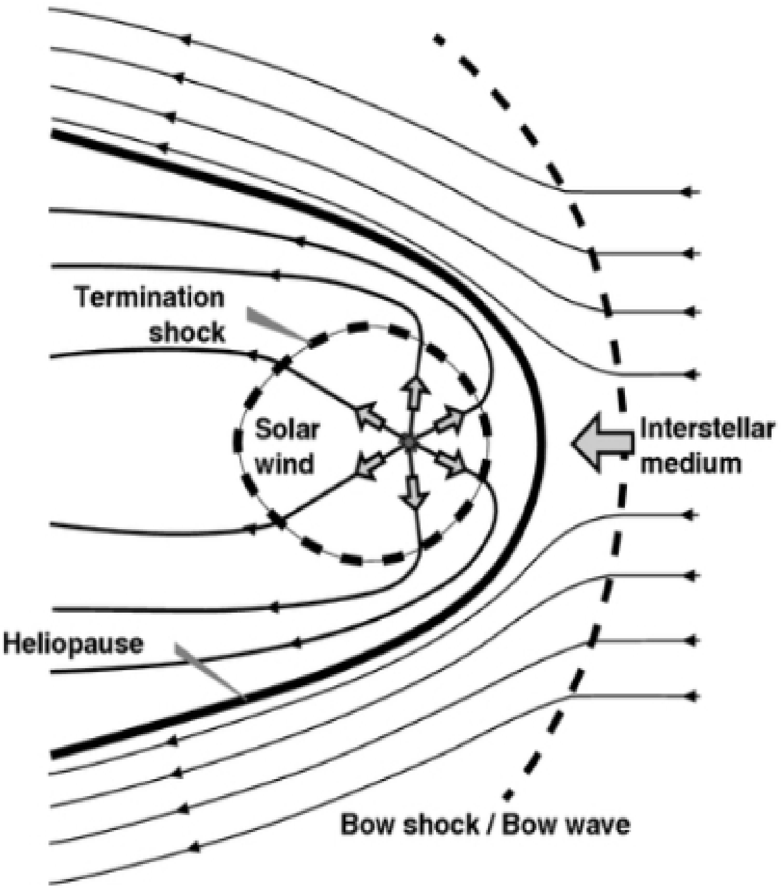
* coronal mass ejections, CMEs, are the main source of impulsive IP shocks in the inner heliosphere

* if the CME velocity is supersonic relative to the ambient solar wind, then as it expands and evolves it is preceded by an IP shock

* arrival of the IP shock to Earth marks the start of terrestrial effects from the CME



Example of collisionless shock: solar wind termination shock



* the outward expansion of the solar wind is eventually halted by momentum balance with the local interstellar medium, ISM, at ~100 AU from the Sun

* the boundary between the solar plasma and the plasma from ISM is called **heliopause**

* the solar wind is supersonic relative to the heliopause, so interior to the heliopause an inner standing shock forms, which is called **termination shock**

* the region of subsonic flow between the termination shock and the heliopause is called the **heliosheath**

* termination shock may not be the most distant heliospheric shock: the Sun is moving relative to the local ISM at a speed believed possibly supersonic, so that a giant bow shock might form exterior to the heliosphere in the flow of the ISM plasma

MHD discontinuities and shocks

* the action of magnetic field tends to replace the role of collisions in “binding” the particles of the plasma together

* MHD does not include effects of the individual particles, it cannot explain how the shock provides dissipation, nor what the structure of the shock will be

* MHD is suitable to describe the plasma upstream and downstream of the shock itself

* in the single fluid case of collisionless plasma Rankine-Hugoniot relations provide a unique prescription for the downstream state

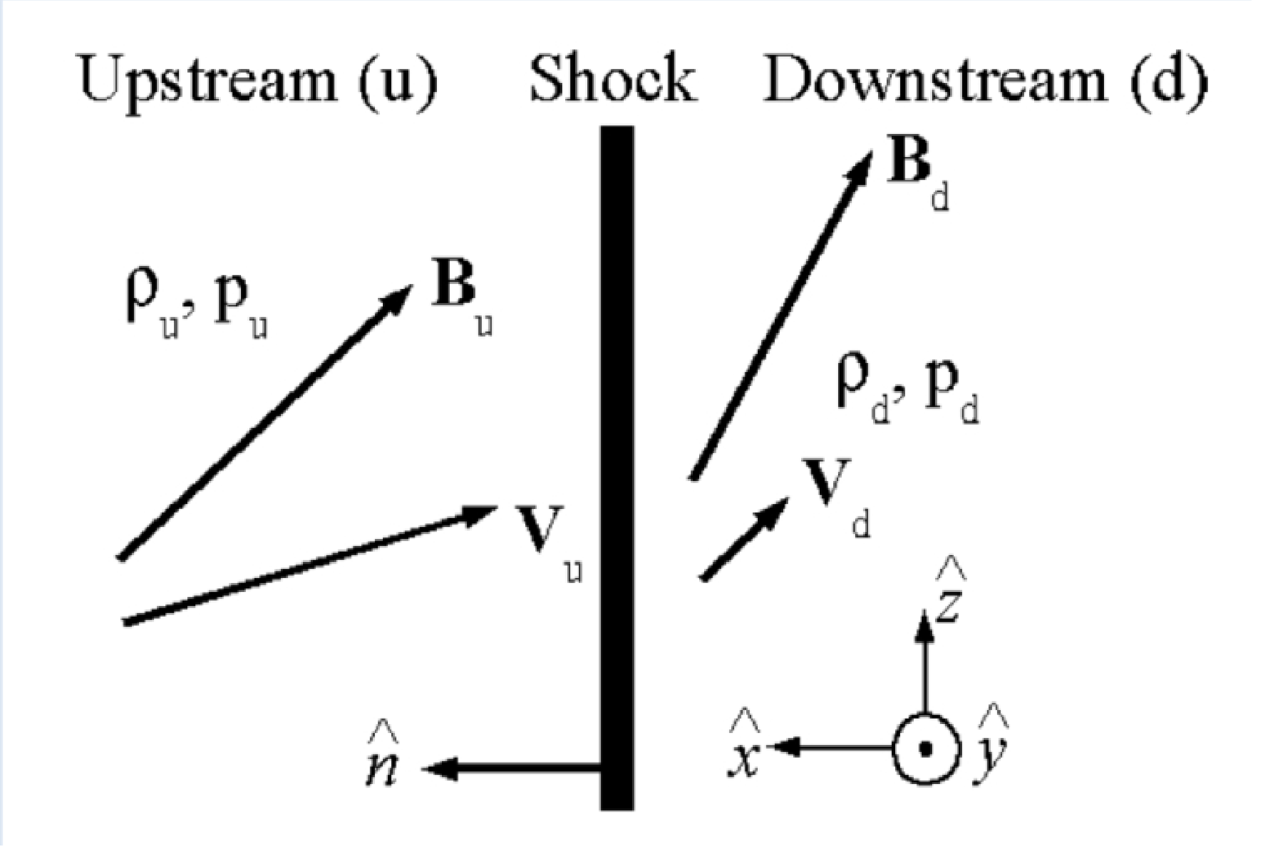
* in two fluid case Rankine-Hugoniot relations do not provide a unique solution, mainly because energy conservation only gives information about the total pressure, and not how it is divided between amongst the different types of particles in plasma

The MHD Rankine-Hugoniot relations

* assumptions: the shock is stationary in the shock frame, the energy in waves is not important, the particle distributions can be described by Maxwellians, plasma pressure is isotropic

* one of the basic observational tests for a shock is a comparison with the Rankine-Hugoniot relations

* a consider a one-dimensional steady shock is considered



* for a steady planar discontinuity a set of jump conditions can be derived

The MHD Rankine-Hugoniot relations

- MHD conservation relations lead to complicated set of different types of discontinuity

- Evolutionary conditions complicated by multiple wave modes and fact that two sets of conditions have to be solved (Alfvenic perturbations decouple from magnetosonic and entropy)

- For shocks (with mass flux across surface) only FAST and SLOW mode shocks are evolutionary

  FAST: $V_{n,u} > V_{f,u}$ and $V_{f,d} > V_{n,d} > V_{A,d}$

  SLOW: $V_{A,u} > V_{n,u} > V_{s,u}$ and $V_{s,d} > V_{n,d}$

- Strong shocks are almost definitely fast mode shocks!

$$[\rho V_x] = 0$$

$$\left[\rho V_x^2 + p + \frac{B^2}{2\mu_0}\right] = 0$$

$$\left[\rho V_x \mathbf{V}_t - \frac{B_x}{\mu_0}\mathbf{B}_t\right] = 0$$

$$\left[\rho V_x \left(\frac{1}{2}V^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\right) + V_x \frac{B^2}{\mu_0} - \mathbf{V}\cdot\mathbf{B}\frac{B_n}{\mu_0}\right] = 0$$

$$[B_x] = 0$$

$$[V_x \mathbf{B}_t - B_x \mathbf{V}_t] = 0$$

-> the solutions of these jump equations describe a number of different types of MHD discontinuity, including shocks

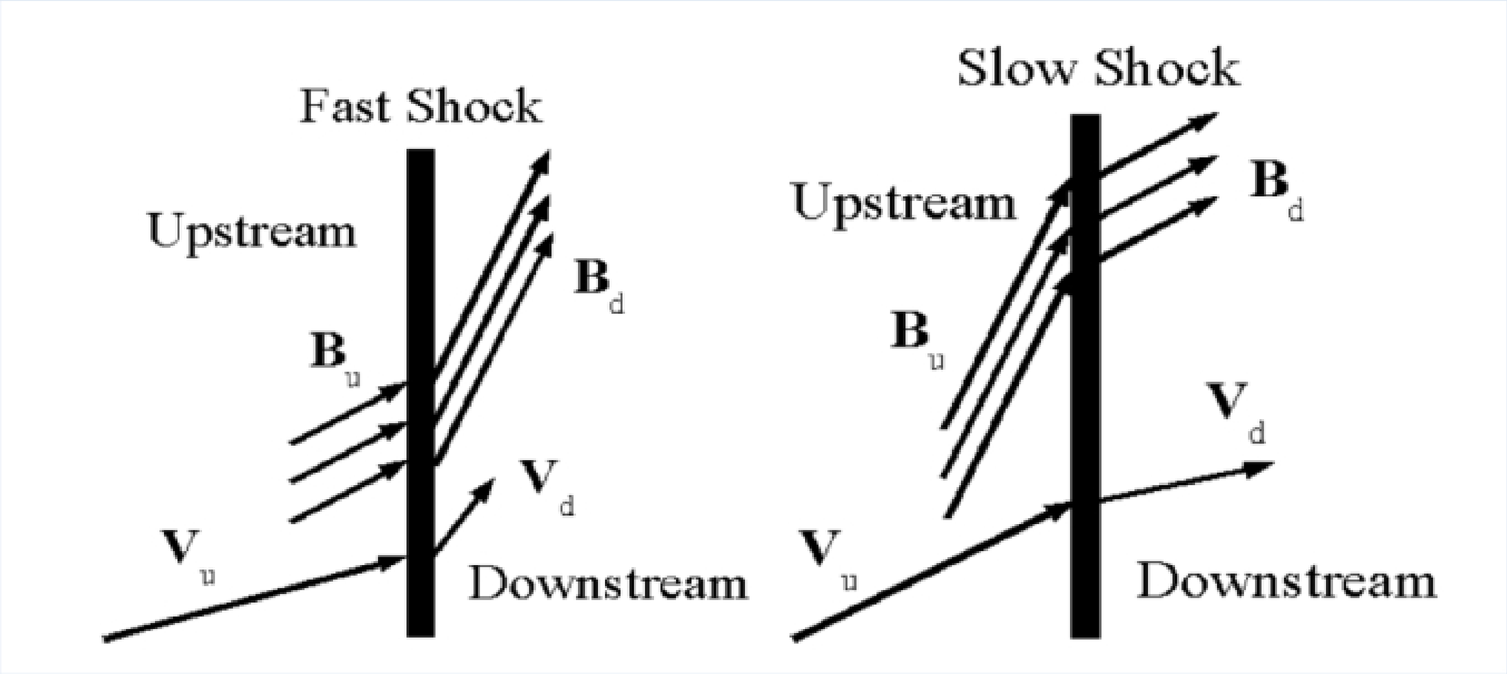
Discontinuities

Contact Discontinuity	$\mathbf{V}_u = \mathbf{0}$, $B_n \neq 0$	Density jump arbitrary, but pressure and all other quantities are continuous.
Tangential Discontinuity	$V_x = 0$, $B_n = 0$	Plasma pressure and field change maintaining static pressure balance.
Rotational Discontinuity	$V_n = B_n/\sqrt{\mu_0 \rho}$	Form of intermediate shock in isotropic plasma, field and flow change direction but not magnitude.

Shock Waves: $\mathbf{V}_u \neq \mathbf{0}$

Parallel Shock	$B_t = 0$	Magnetic field unchanged by shock.
Perpendicular Shock	$B_n = 0$	Plasma pressure and field strength increases at shock.
Oblique Shocks	$B_t \neq 0$, $B_n \neq 0$	
Fast Shock		Plasma pressure and field strength increase at shock, magnetic field bends away from normal.
Slow Shock		Plasma pressure increases, magnetic field strength decreases, magnetic field bends towards normal.
Intermediate Shock		Only shocklike in anisotropic plasma.

Refraction of magnetic field



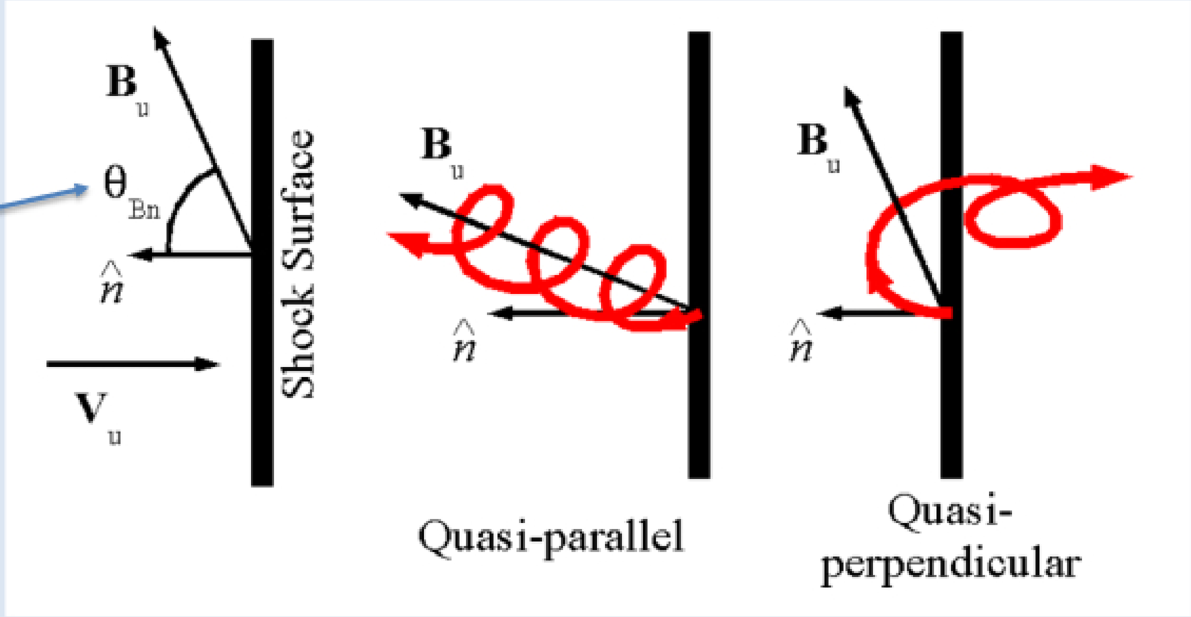
$V_{n,u} > V_{f,u}$ and $V_{f,d} > V_{n,d} > V_{A,d}$

$V_{A,u} > V_{n,u} > V_{s,u}$ and $V_{s,d} > V_{n,d}$

Shock parameters

Shock normal angle

θ_{Bn}



Alfvén Mach Number $M_A = \frac{V_u}{B_u/\sqrt{\mu_0 \rho_u}}$

Other plasma parameters: plasma beta, composition, anisotropy, etc!

The exactly parallel shock

$$\mathbf{B}_u = B_x\hat{n},\ \mathbf{B}_{ut} = \mathbf{0}$$

Using transverse momentum equation

$$\left[\left(1 - \frac{B_n^2}{\mu_0 \rho V_n^2}\right) V_n \mathbf{B}_t\right] = \mathbf{0}$$

Transverse magnetic field zero upstream and downstream, so magnetic field is unchanged by shock.

Shock jump relations same as for an ordinary gas shock.

The exactly perpendicular shock

$B_x = 0$ and $\mathbf{B}_u = \mathbf{B}_{ut}$

From conservation of tangential electric field

$$V_{ux}\mathbf{B}_{ut} = V_{dx}\mathbf{B}_{dt}$$

density *compression ratio* $r = \rho_d / \rho_u$

Magnetic field compresses as much as the density

For high Mach number, compression ratio becomes

$$r = \frac{\gamma + 1}{\gamma - 1}$$

$M_A \gg 1$ and $M_{cs} \gg 1$

Compression ratio has limit of about 4 as Mach number increases

Oblique shocks

* upstream field is neither exactly parallel nor perpendicular to the shock normal

* it is always possible to move to a frame of reference in which the shock is at rest and the upstream velocity is directed into the shock along the shock normal: the “Normal Incidence Frame”, NIF

Coplanarity Theorem: At oblique MHD shocks, the shock normal $\hat{n}$, upstream and downstream magnetic fields $\mathbf{B}_u$ and $\mathbf{B}_d$, and jump in velocity $\mathbf{V}_u - \mathbf{V}_d$ are all coplanar.

$$\hat{n} = \frac{(\mathbf{B}_u \times \mathbf{B}_d) \times (\mathbf{B}_u - \mathbf{B}_d)}{|(\mathbf{B}_u \times \mathbf{B}_d) \times (\mathbf{B}_u - \mathbf{B}_d)|} \qquad V_n = \frac{B_n}{\sqrt{\mu_0 \rho}} \equiv v_{An}$$

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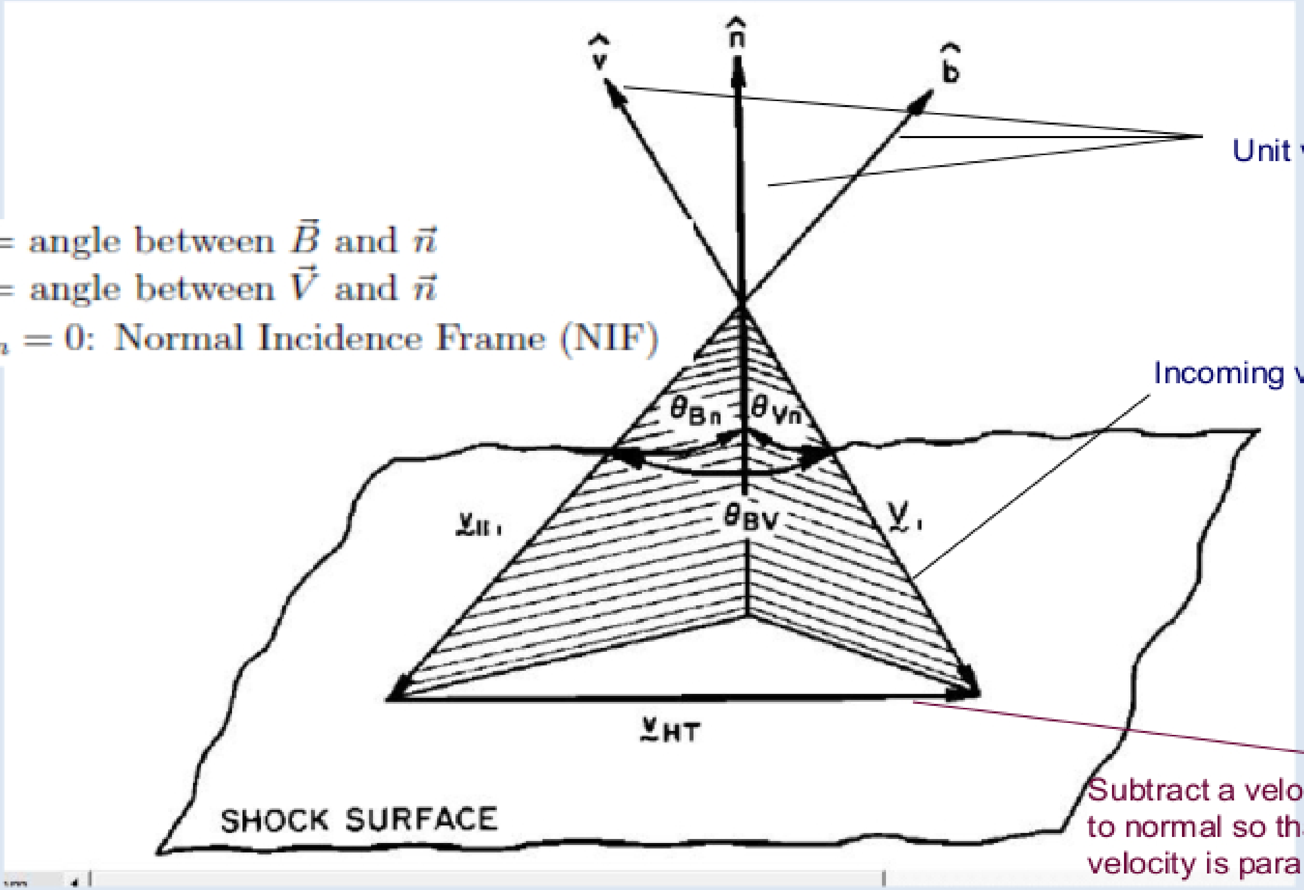
$$V_n = \frac{B_n}{\sqrt{\mu_0 \rho}} \equiv v_{An}$$

De Hoffman-Teller frame, HTF

θ_{Bn} = angle between $\vec{B}$ and $\vec{n}$

θ_{Vn} = angle between $\vec{V}$ and $\vec{n}$

If $\theta_{Vn} = 0$: Normal Incidence Frame (NIF)



De Hoffman-Teller frame

The de Hoffmann-Teller velocity

$\vec{V}_{in}$ = incoming velocity, $\vec{V}_{HT}$ = de Hoffmann-Teller velocity, $\vec{V}^*_{in}$ = incoming velocity in the H-T frame

$$\vec{V}_{in} = \vec{V}_{HT} + \vec{V}^*_{in}$$

Take cross-product with $\vec{B}$

$$\vec{B} \times \vec{V}^*_{in} = 0 = \vec{B} \times \vec{V}_{in} - \vec{B} \times \vec{V}_{HT}$$

take cross-product with $\vec{n}$

$$\vec{n} \times (\vec{B} \times \vec{V}_{HT}) = \vec{n} \times (\vec{B} \times \vec{V}_{in})$$

$$\vec{n} \times (\vec{B} \times \vec{V}_{HT}) = \vec{B}(\vec{n} \cdot \vec{V}_{HT}) - \vec{V}_{HT}(\vec{n} \cdot \vec{B})$$

$$\vec{V}_{HT} = -\frac{\vec{n} \times (\vec{B} \times \vec{V}_{in})}{\vec{n} \cdot \vec{B}}$$

De Hoffman-Teller frame

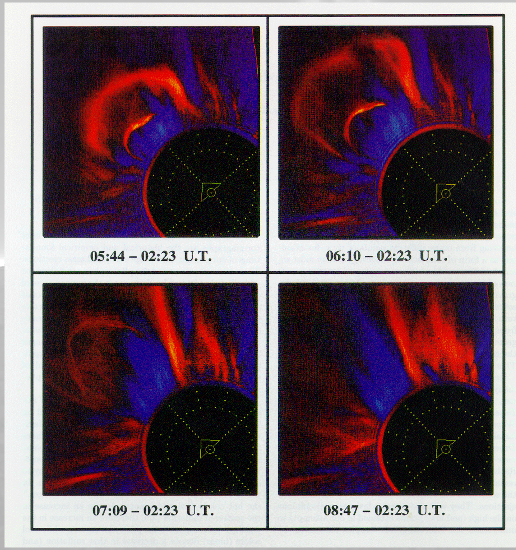
1. Since in the de Hoffmann -Teller (HT) frame the velocity is parallel to the magnetic field the $\vec{V} \times \vec{B}$ electric field vanishes in that frame

2. In the HT frame the shock is at rest - no induced electric field

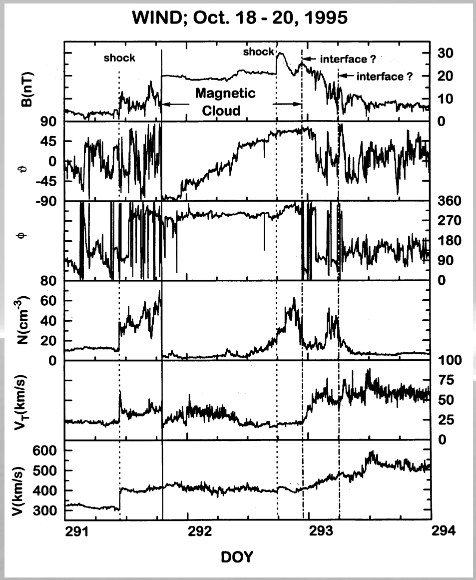
A consequence of $\vec{E} = 0$ is that the energy of a particle is constant, and surfaces of constant particle energy are spheres centered on the HT frame origin

3. The de Hoffmann-Teller transformation velocity is the same downstream as it is upstream

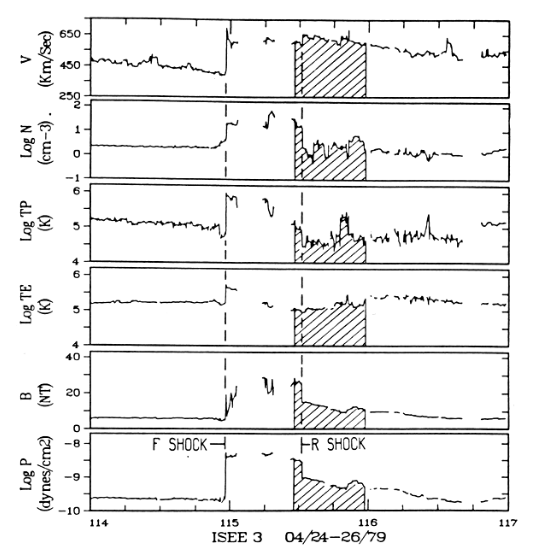
Observed shocks



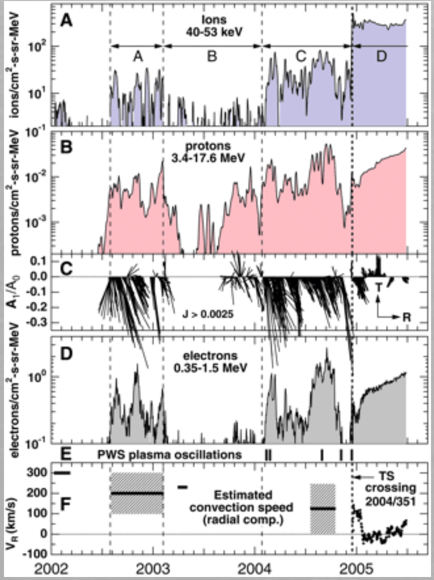
CME



Magnetic cloud



Corotating Interaction Region



The Termination Shock

Shock structure: real shocks

* direct observations of naturally occurring shocks in space

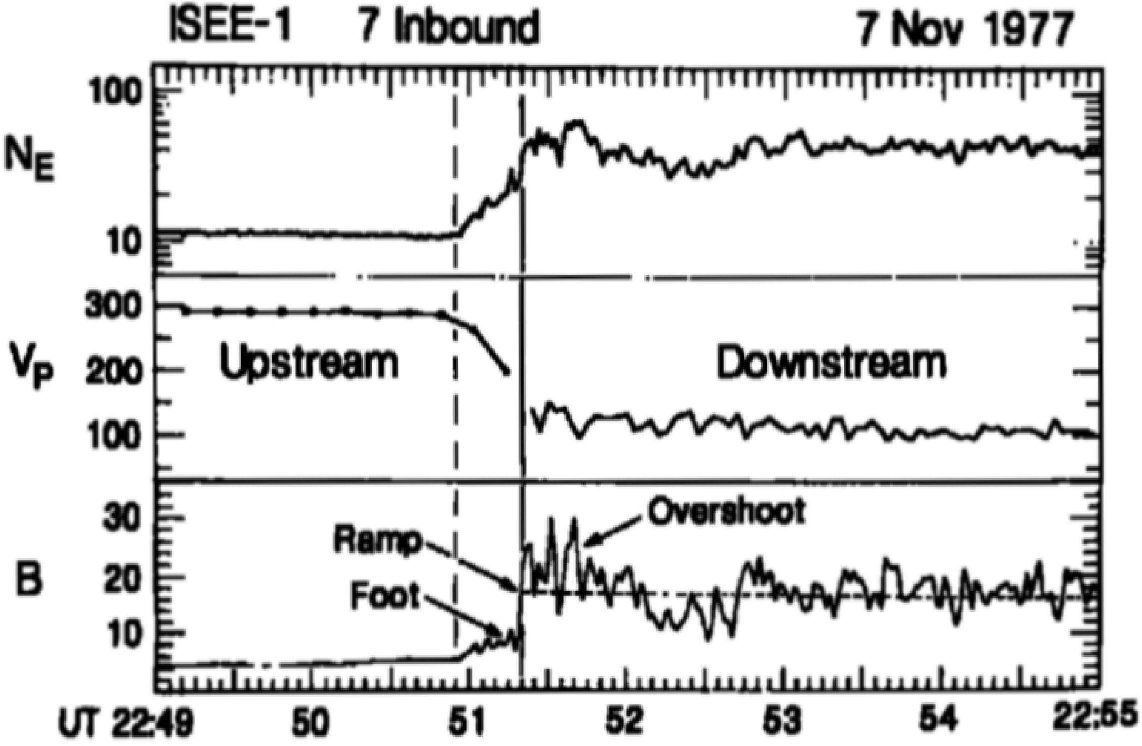
* collisionless shocks in the laboratory cannot approach the scale of the naturally occurring space shocks

* space observations are also unique in that the smallest plasma scale, the Debye length is usually larger than the spacecraft -- we can truly make a point measurement, since our measurement devices do not affect the plasma

* high resolution measurements of the space shocks, measurements of how the plasma changes as it passes through the shock, the collisionless dissipation mechanisms in action

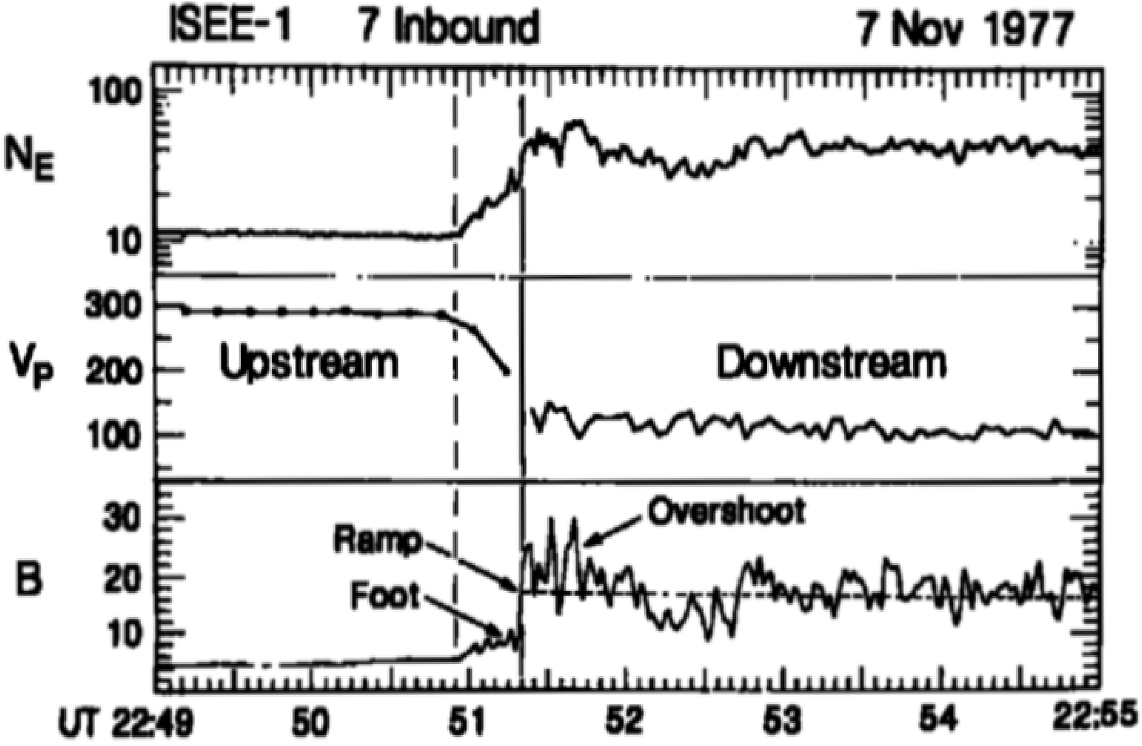
Shock structure: real shocks

* measurements made as a spacecraft passed through the Earth's bow shock
* instruments onboard measure the magnetic field
* by counting particles arriving at the satellite, the electron and proton distribution functions can be measured -> the density and the flow speed of the solar wind as it passes through the shock can be calculated
* the satellite has a very small speed, and usually observations of the bow shock happen when the shock moves across the satellite due to slight changes in the solar wind speed or density
* if the speed of the shock relative to the satellite is constant, then the profiles we observe in time will be the same as the shock's profile in space
* the satellite is initially in the solar wind, and then the shock moves outwards and the observations are taken as the



Shock structure: real shocks

* in passing from upstream to downstream it is obvious that the velocity decreases, and the density increases – there is compression at the shock.
* the magnetic field increases like the density – it is a fast mode shock
* the shock is thin, but it is not just a smooth transition
* structure within the transition: a “foot”, a “ramp” and an “overshoot” which are controlled by the way in which the solar wind protons heat at the shock
* the thickness of the foot equals the distance to which protons are reflected by the shock before drifting through it



Different parameters make different shocks

* the most important result from space observations of shocks is that there are many different types of shock depending on the shock parameters

* the Rankine-Hugoniot relations lead to different modes of shocks

* even if restricting ourselves to fast mode shocks, like planetary bow shocks, we find different types of shock structure

* early on there were quite serious debates about whether the Earth's bow shock was actually stable; perhaps all the different profiles seen in observations were just fleeting glimpses of an ever changing entity

* a major contribution was to show that there was a definite pattern to the observations, and that the type of shock was determined by the complete set of shock parameters describing the upstream conditions

* the most important factors in controlling the type of shock are the direction of the upstream magnetic field, relative to the shock surface, and the strength

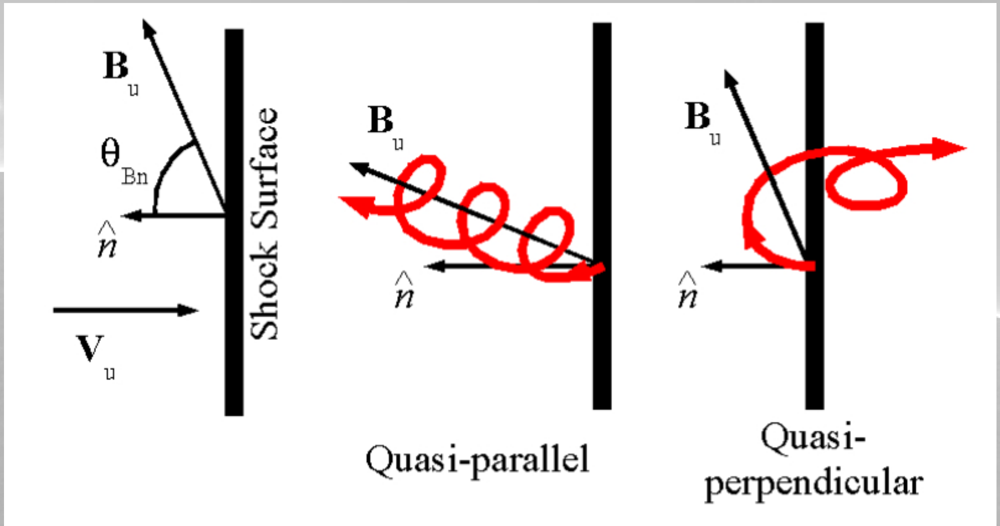
Different parameters make different shocks

* influence of the direction of the upstream field called the shock geometry

* a one-dimensional shock, marked on it is the unit vector $\hat{n}$ normal to the shock Surface

* depending on θ_{Bn} the shock has dramatically different behaviors

* we consider the motion of a particle which is initially headed away from the shock



* shock geometry θ_{Bn} (left) and the motion of reflected ions in the case of quasi-parallel geometries (middle) and quasi-perpendicular ones (right)

* such particles arise either by escaping upstream from the downstream side or by being reflected at the shock itself

Parameters for describing particle motion: pitch angle

* pitch angle is the angle between the velocity vector **v** and the magnetic field **B**:

$$\alpha = \tan^{-1} \frac{v_\perp}{v_\parallel}$$

* $v_\parallel = v \cos \alpha$ and $v_\perp = v \sin \alpha$

* particles with small pitch angle are called field aligned and those with large, i.e. close to 90º, pitch angle are sometimes called gyrating

Parameters for describing particle motion: guiding centre

* particle motion is often decomposed into drifts plus gyration, because often the gyration is on a scale much smaller than the physical system and much more rapid than the phenomena of interest

* motion of the average centre of gyration, known as the guiding centre

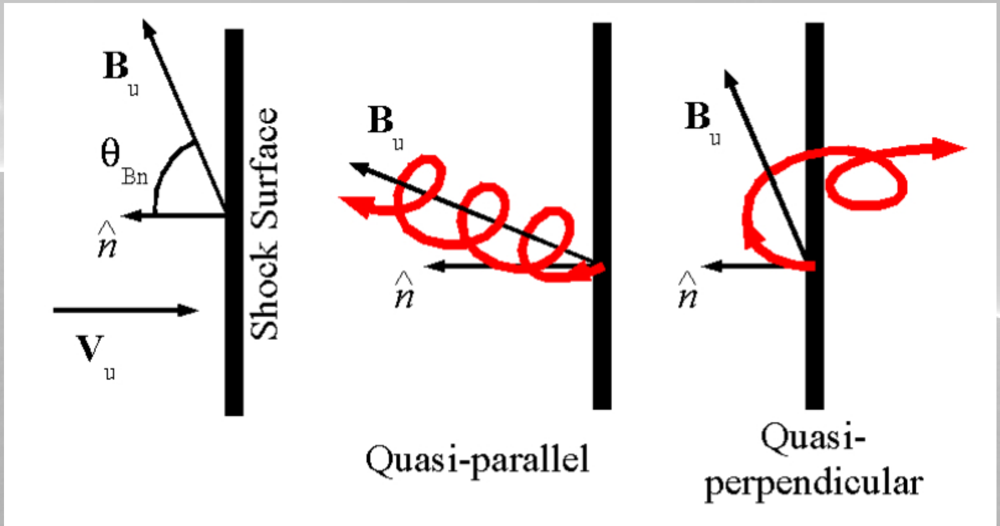
* for “slow” changes in the field the motion of the guiding centre is continuous

* a way to calculate the guiding centre position is to average the particle position over one cyclotron period, i.e. the time taken to complete one gyration

Different parameters make different shocks

* the motion of particles initially headed away from the shock is particularly simple if we transform away the motional electric field by moving along the shock surface, so that the upstream velocity V_u is aligned with the magnetic field B_u

* in this frame, de Hoffmann-Teller frame, the shock is still at rest and now the particle motion is pure gyromotion plus field-aligned guiding centre motion, since $\mathbf{E'_0} = -\mathbf{V'_0} \times \mathbf{B_0} \equiv 0$



* shock geometry θ_{Bn} (left) and the motion of reflected ions in the case of quasi-parallel geometries (middle) and quasi-perpendicular ones (right)

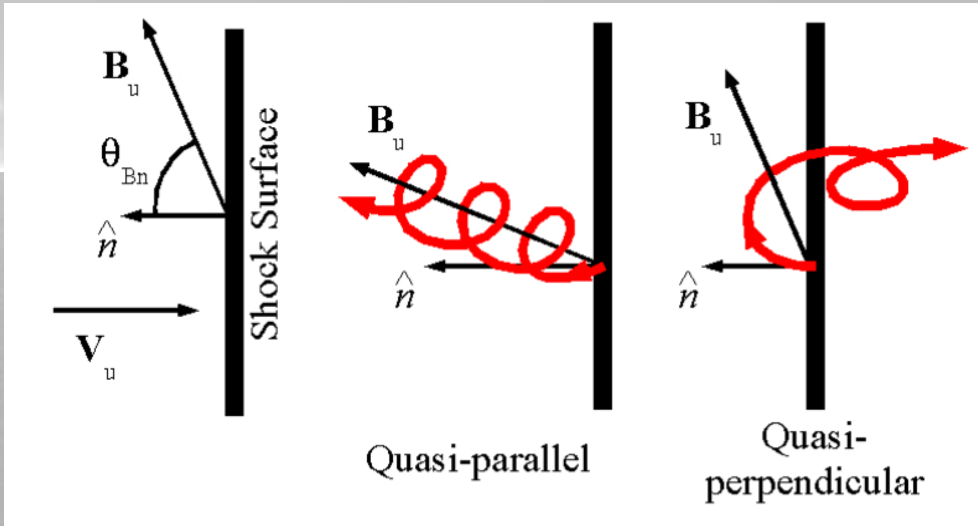
* in the case of a quasi-parallel shock the field lines pass through the shock and a particle's motion along the field will carry the particle through and away from the shock relatively easily

Different parameters make different shocks

* at quasi-perpendicular shocks the field lines are almost parallel to the shock surface, so particle motion along the magnetic field would need to be very rapid, i.e. the pitch angle would need to be quite small, if the particle is to escape from the shock

* for most pitch angles, the particle gyration at a quasi-perpendicular shock brings the particle back to the shock

* particle motion in the normal direction is “easier” at the parallel shock, compared to the perpendicular shock – expectation that the scale of the parallel shock will be larger than for the perpendicular shock



* shock geometry θ_{Bn} (left) and the motion of reflected ions in the case of quasi-parallel geometries (middle) and quasi-perpendicular ones (right)

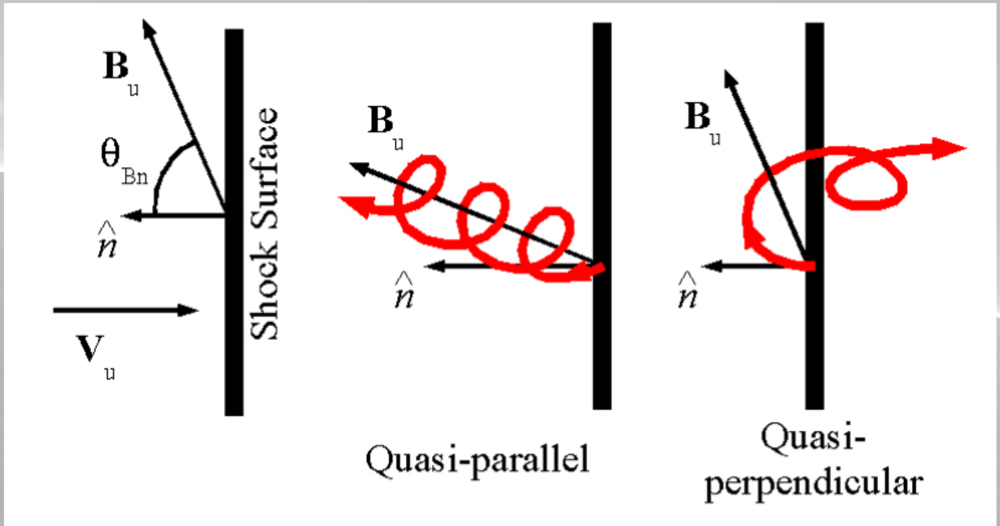
* ions have much larger gyroradii than the electrons, they determine most of the structure of collisionless shocks, because of their much greater mass the ions carry most of the mass and energy flow in a plasma

Different parameters make different shocks

* the shock angle θ_{Bn} is the most important factor in controlling the shock type, but almost every plasma parameter can have an effect: temperature, composition and shock Mach number M

* the Mach number indicates the strength of the shock, and is a measure of the amount of energy being processed by the shock: the higher the Mach number the more dramatic the behavior of the shock

* in the solar system shocks can be found with Mach number between 1 and 20



* shock geometry θ_{Bn} (left) and the motion of reflected ions in the case of quasi-parallel geometries (middle) and quasi-perpendicular ones (right)

* in the shocks produced by super-nova remnants the Mach number could be of the order of thousand

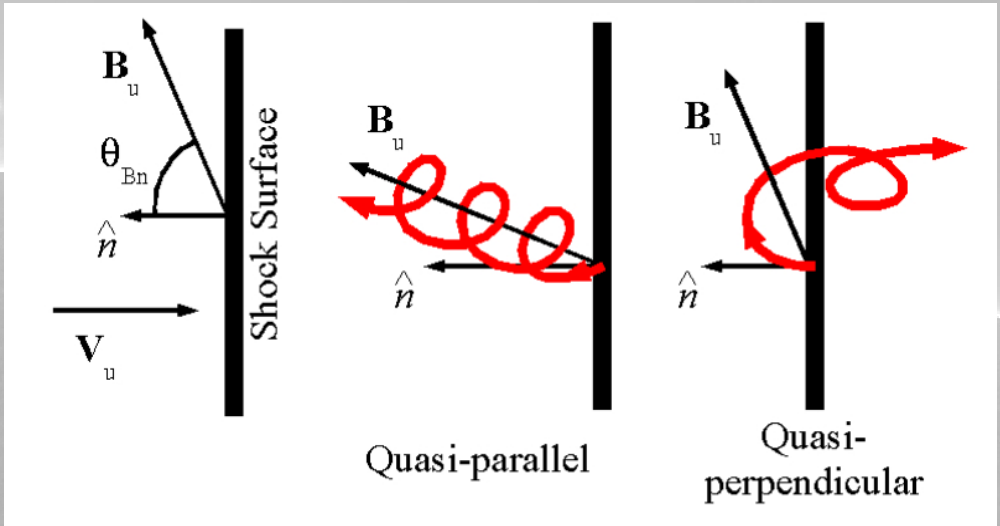
* for the solar system quasiperpendicular shocks a fairly good understanding

Different parameters make different shocks

* in quasi-parallel shocks the easy particle access results in an extended region of waves, turbulence, and shock-energised suprathermal particles

* in the quasi-perpendicular shock there is a clear distinction between a type of low Mach number shock, the subcritical shock, and a higher Mach number shock, the supercritical shock

* for most of the time the Earth's bow shock is supercritical, and to find subcritical shocks we usually have to wait for a suitable interplanetary shock



* shock geometry θ_{Bn} (left) and the motion of reflected ions in the case of quasi-parallel geometries (middle) and quasi-perpendicular ones (right)

* a "critical" Mach number which separates the two types of shock: at $M_A \sim 2.7$ for $\theta_{Bn} = 90^\circ$ and decreases as θ_{Bn} decreases

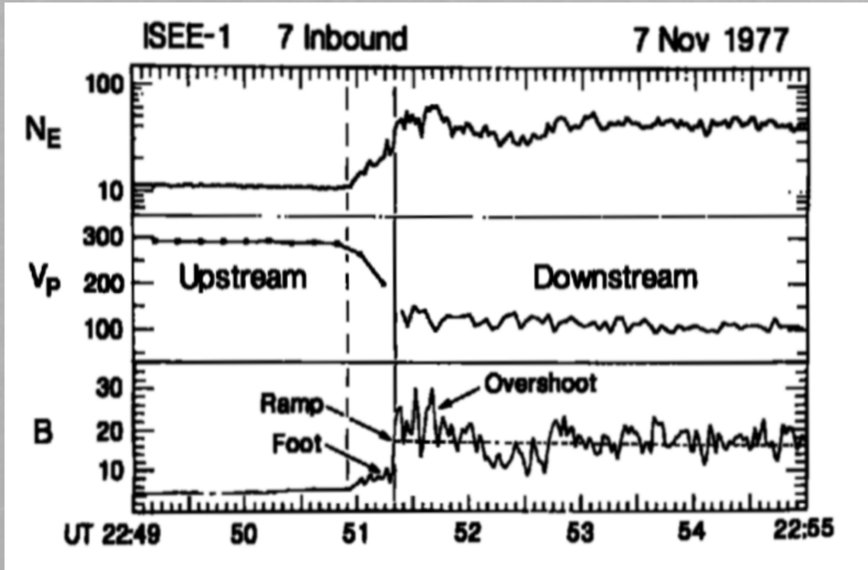
* the Earth's bow shock has values for M_A in the range $\sim 1.5 - 10$

Different parameters make different shocks

* this is example is of a supercritical shock

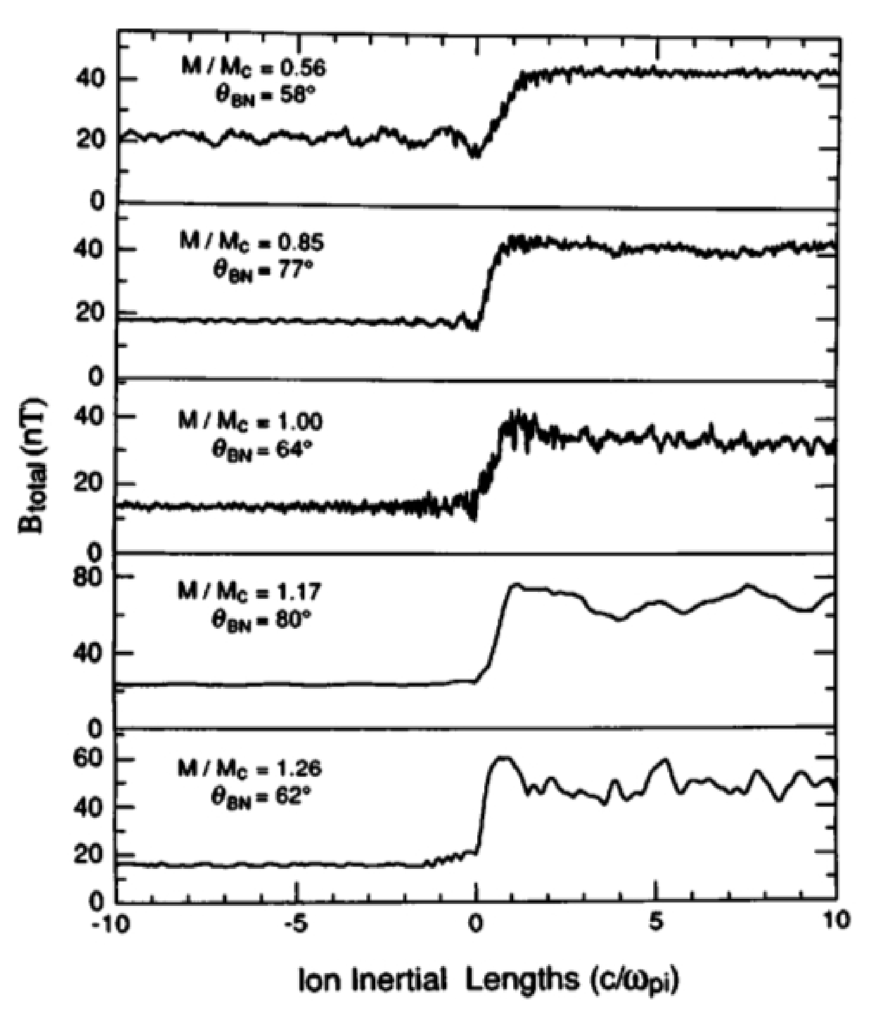
* visible structure: – the field has a single sharp jump called the **ramp**, which is preceded by a gradual rise called the **foot**, which is related directly to the gyrating orbits of ions reflected by the shock and returned to the shock surface as sketched in the right panel of the previous slide; the field right at and behind the ramp is higher than its eventual downstream value, and this is called the **overshoot**

* the small wiggles in the field are small amplitude turbulence, which may play a role, but doesn't control the basic shock structure



Different parameters make different shocks

* the subcritical shock resembles much more the profile of a shock in an ordinary gas
* several examples of low Mach number shocks ranging from subcritical to slightly supercritical
* the top two have relatively simple ramps from the upstream to the downstream value, with little or no foot or overshoot
* a wave appears in the upstream region, the **precursor**, with a single well-defined wavelength; it depends on both the θ_{Bn} and the Mach number; it corresponds to non-MHD aspects of shorter wavelength waves which travel faster than the fast magnetosonic wave and hence can escape upstream
* the wavelength is smaller in the second, more perpendicular, of these examples
* as supercriticality is reached and surpassed, the foot and overshoot become apparent
* the size, shape, and extent of the shock features



Diffusive shock acceleration

* a shock, travelling through a medium in which particles are diffusing, can accelerate some of those particles to high energies

* in the steady state the acceleration process results in power-law energy spectra

* the process relies on multiple scattering of the particles in the upstream and downstream medium and was thus also termed first-order Fermi acceleration.

* the diffusive shock acceleration theory first derived from the Parker transport equation of energetic particles in a magnetized plasma: the equation describes spatial transport of energetic particles by convection of the background plasma and by spatial diffusion, and momentum changes due to compression or expansion of the background plasma

* the diffusive acceleration process has been thoroughly investigated theoretically: coupling of the particles to the self-excited waves, time dependence of the acceleration process, spatial dependence of the diffusion coefficient, limited size of the acceleration region, and shock modification by the accelerated particles have been analysed

Diffusive shock acceleration

* all shocks observed either in situ or indirectly in the solar system are accompanied by energetic particles, i.e. protons, heavy ions and electrons, with energies up to 1000 MeV

* the most intense solar energetic particle, SEP, events are produced by acceleration at interplanetary shock waves driven by coronal mass ejections, CMEs

* the intensity–time profiles of electrons and protons in CME associated particle events have usually a fast rise and a decay phase extending over several days.

* the shock passage at 1 AU occurs early in the decay phase and is often accompanied by a peak in the lower-energy ions (MeV).

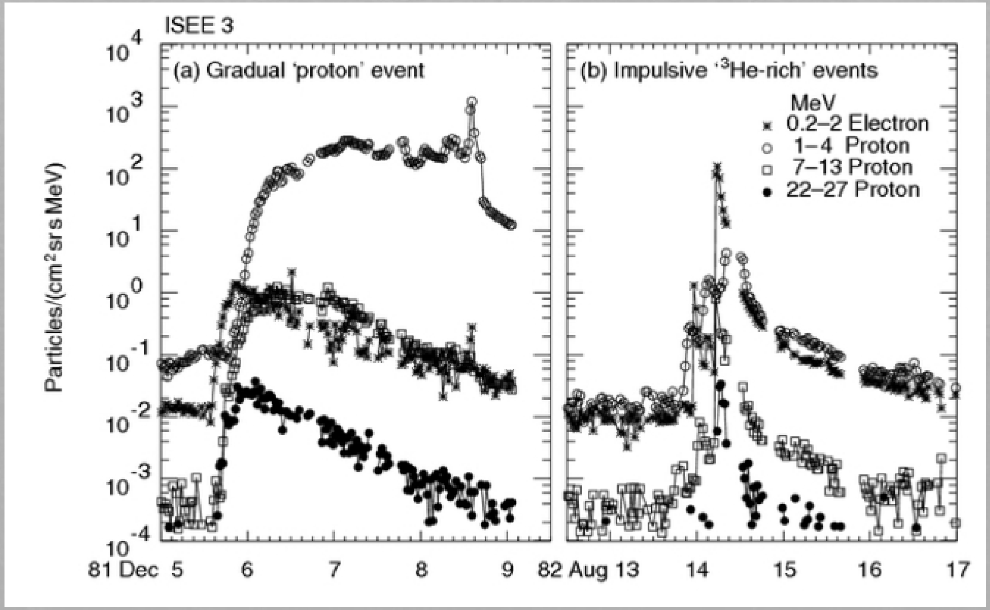
* these events are long-lasting, several days, they have been termed **gradual events**, as opposed to **impulsive events** of a duration of approximately a few hours.

* the impulsive events are associated with impulsive X-ray flares and Type-III radio bursts. X-ray emission from flares indicates plasma heating to temperatures of order ten million kelvin, and solar Type-III radio bursts imply the impulsive production of electrons travelling at 0.1–0.25 times the speed of light.

Diffusive shock acceleration

* acceleration at the CME driven shocks populate magnetic field lines over a broad range of longitudes, while the impulsive events are generally detected when the observer is magnetically connected to the flare site

* figure (a) shows the intensity–time profiles of particle fluxes (one electron and three proton energy channels) in a ‘pure’ gradual event accompanied by a CME and (b) shows an impulsive event due to a series of flares

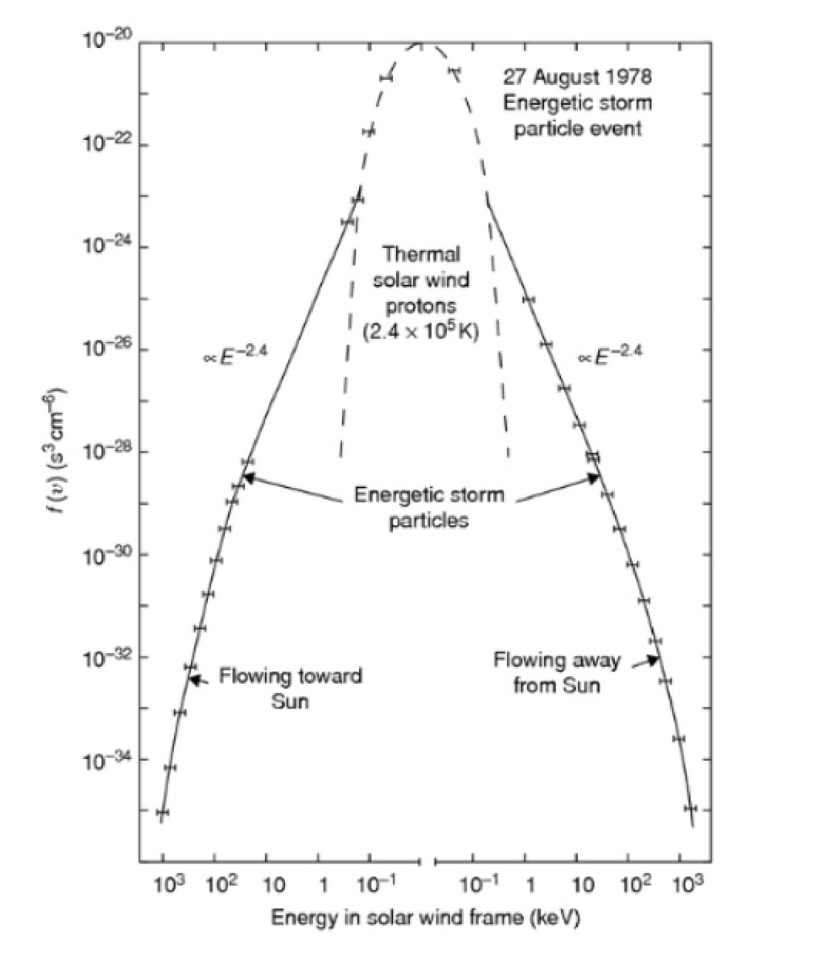


Diffusive shock acceleration

* evidence that energetic ions are accelerated at interplanetary shocks out of the solar wind thermal population

* the ion distribution function measured over a wide energy range, 10 eV to 1.6 MeV, behind an interplanetary shock is transformed into the solar wind frame

* figure presents a cut of the distribution function along the Sun–Earth line, which demonstrates that the suprathermal distribution emerges smoothly out of the solar wind thermal distribution -- evidence that the solar wind ions are accelerated to high energies.



The distribution function of ions from 10 eV to 1.6 MeV in the solar wind frame of reference during the post-shock phase of a solar energetic particle event.

Diffusive shock acceleration

* this simple viewpoint has been challenged by recent detailed comparison of the ionic charge state, elemental and isotopic composition of solar energetic particle events over a wide energy range with the corresponding solar wind composition: in gradual events at 5 MeV/nucleon the event-integrated Fe/O ratio varies by an order of magnitude; at MeV/nucleon this ratio varies by 3 orders of magnitude

* in some events the energy spectra are well fit by power laws up to the highest energies, but in other events the spectra are power laws with exponential rollovers at high energies

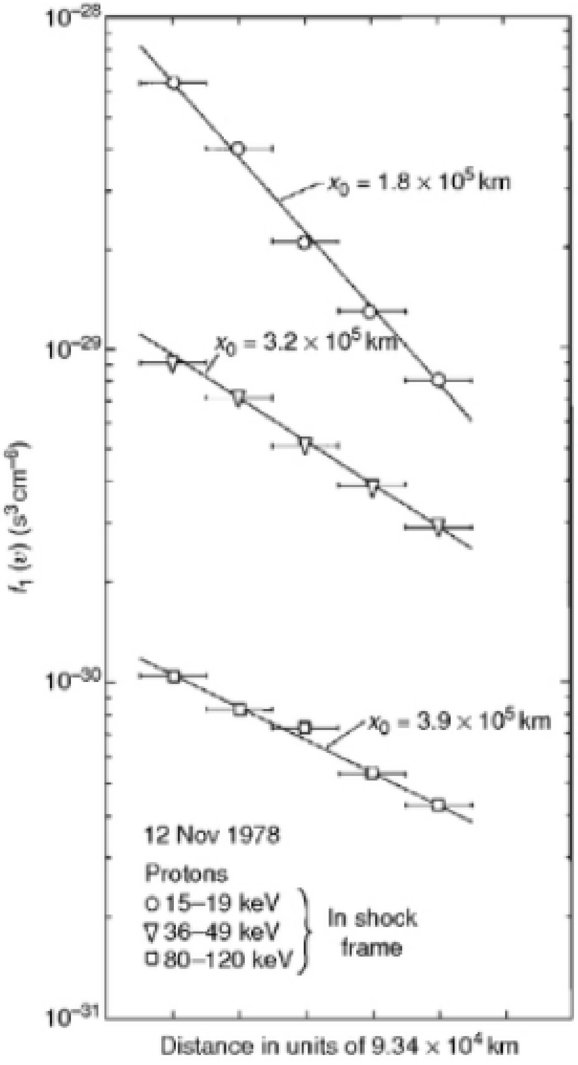
* this indicates the importance of suprathermal particles as seed particles in the coronal / interplanetary shock acceleration process and the possible modification of the abundances by the shock geometry (quasi-parallel versus quasi-perpendicular shock)

* upstream of interplanetary shocks the intensity in the lower-energy range begins to rise several hours before the shock passage

Diffusive shock acceleration

* figure shows the distribution function versus time elapsed between the sampling time and the shock arrival time

* the time interval has been converted into distance from the shock, where a local shock speed was obtained from mass flux conservation



The distribution function of protons at three different energies (in the shock frame) versus time elapsed between sampling and shock passage. Time has been converted into distance from shock. x_o is the e-folding distance obtained from an exponential fit through the data.

Diffusive shock acceleration

* the clearest evidence for particle acceleration at collisionless shocks comes from observations at the Earth’s bow shock, where the low relative velocity between the spacecraft and the shock allows a detailed view of the shock structure and of the accelerated particles

* the bow shock has a relatively small radius of curvature, which means that different regions may interact: ions backstreaming from the quasi-perpendicular region can be swept by the solar wind into the quasi-parallel bow shock, and may contribute as seed particles or influence the acceleration process via excitation of waves and turbulence

* the diffuse ion density falls off exponentially away from the shock along the magnetic field

* multi spacecraft mission Cluster allowed determination of the density gradient when the inter-spacecraft separation was relatively large and when the solar wind stayed relatively constant

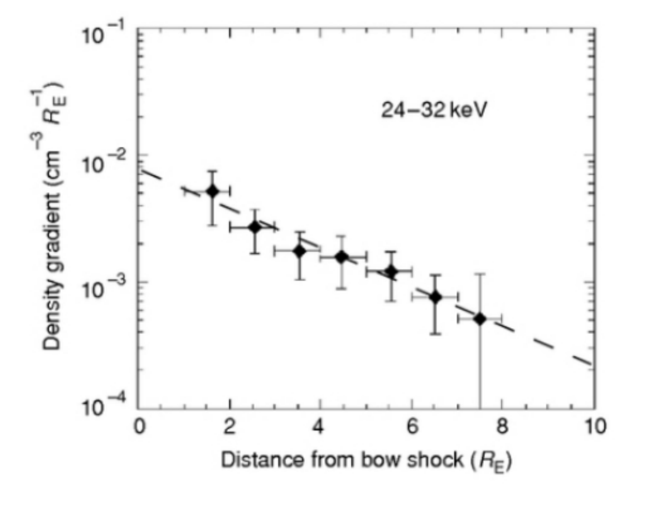
Diffusive shock acceleration

* figure shows log of the measured density gradient as a function of distance from the bow shock for one energy channel

* the linear relationship in that figure confirms that the density profile is exponential as a function of distance from the shock

* this has important consequences for the transport of diffuse ions in the upstream region: independent of the acceleration mechanism at the shock an exponential density decrease along the magnetic field in the upstream direction shows that the convection of the ions with the solar wind toward the shock is balanced by <u>upstream diffusion</u>

* energetic ion transport in the region upstream of the quasi-parallel shock is diffusive



* average partial ion density gradient in the 24–32 keV energy range versus distance from the bow shock

Diffusive shock acceleration

* IP shocks may form as the bounding edges of corotating interaction regions, CIRs

* these shocks create a stationary pattern in the frame corotating with the Sun

* observations of the interaction of fast and slow solar wind streams show that at distances of several astronomical units these CIR associated shocks are accompanied by energetic particles

* the interaction of the solar wind with the local interstellar medium causes the formation of the heliospheric termination shock, the heliopause and possibly a heliospheric bow shock

* the termination shock marks the transition of the solar wind from a supersonic speed to a subsonic speed, and has been crossed in 2004 by Voyager 1 at a distance of 94.0 AU and in 2007 by Voyager 2 at a distance of 83.6 AU, respectively.

* both Voyager spacecraft have observed energetic protons and electrons upstream of the termination shock and in the heliosheath

Coherent shock acceleration

* a particle has a single ‘interaction’ with the shock during which it may cross the shock transition multiple times but remains within approximately a gyrodiameter of the shock

* after the interaction the particle may exit the shock region either upstream or downstream, and (for a particle initially upstream) it will then be classified as ‘reflected’ or ‘transmitted’, respectively

* in the de Hoffmann–Teller frame the motional electric field is zero, since in this frame the flow velocity is parallel to the magnetic field, and, for static shock fields, the particle gains no energy during the interaction

* in the case when there are upstream waves which scatter the particles, diffusive shock acceleration will be present in addition to coherent acceleration at the shock

Electron acceleration

* the evidence from the Earth’s bow shock and other solar system shocks is that collisionless shocks, at least for the observed parameter range, are not characterized by strong electron heating

* collisionless shocks are generally not responsible for strong electron acceleration, either in terms of fluxes or maximum energy.

* observational evidence for some shock acceleration of electrons to moderate energies at the bow shock and at interplanetary shocks

* in both cases there is an electron foreshock populated by superthermal, energized electrons

Simulation techniques

* knowledge about the physics of shocks has to large extent been obtained by the use of numerical simulations: full particle PIC simulations, hybrid simultions and monte Carlo simulations

* a collisionless plasma is described by the Vlasov equation for the distribution function $f_j(\mathbf{x},\mathbf{v},t)$:

$$\frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \frac{\partial f_j}{\partial \mathbf{x}} + \frac{q_j}{m_j}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_j}{\partial \mathbf{v}} = 0,$$

* the **E** and **B** are given by Maxwell's equations, with source terms, i.e. electrical current and charge, determined by the moments of the distribution functions of the total of *M* species:

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 J,$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E},$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho,$$

$$\nabla \cdot \mathbf{B} = 0.$$

$$J = \sum_{j=1}^{M} q_j \int f_j \mathbf{v} d\mathbf{v}$$

$$\rho = \sum_{j=1}^{M} q_j \int f_j d\mathbf{v}.$$

Particle-in-cell method

* fully resolving the phase space requires a vast amount of computer memory and thus the direct approach of solving the Vlasov equation for studying collisionless shocks has been abandoned in favour of representing the distribution functions f_j by a number N of macroparticles

* the orbits of these macroparticles are computed in the self-consistent electric and magnetic fields

* the particle mesh method is used and the particles properties (density, velocity) on a spatial grid are collected anf the field equations are solved

* since simulation particles represent a statistical sampling of the actual distribution function there will be an error on the integrals over the particles, which represents the statistical noise due to finite particle number

Hybrid simulation method

* when it is not necessary to resolve electron spatial and temporal scales, electrons do not have to be treated kinetically – a fluid treatment is sufficient

* examples: the low-frequency waves upstream of quasi-parallel shocks, the interactions of large-amplitude pulsations with quasi-parallel shocks, Alfven ion cyclotron waves downstream of quasi-perpendicular shocks

Methods of shock excitation

* different setups to generate a collisionless shock in a numerical system:

– injection method: numerical box is filled with plasma which streams wit a super-Alfvenic velocity; it reflects from the wall and interacts with the incoming plasma, with time a shock arises

– piston method: the simulation box is divide into one region with vacuum and another with plasma at rest; an external current pulse is applied in the vacuum part which induces a pulse in the electric and magnetic field, which acts as a piston ahead which a shock develops

– the plasma release method: the simulation box is initially divided into one region with a hot plasma which streams into the second region containing a colder and less dense plasma; with time the interface develops into a shock

Monte Carlo simulations of collisionless shocks

* developed for numerical studying of diffusive shock acceleration

* solves the steady state Boltzmann equation for the distribution function in the shock frame

Literature

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