

## Matrices and linear transformations

1. Which of the following transformations are linear?

- a)  $A: (x, y) \mapsto (x, 0)$
- b)  $A: (x, y) \mapsto (0, x)$
- c)  $A: (x, y) \mapsto (1 + x, y)$
- d)  $A: (x, y) \mapsto (2x, y)$
- e)  $A: (x, y) \mapsto (|x|, y)$
- f)  $A: (x, y) \mapsto (x, e^y)$

2. Let  $\vec{a} \neq 0$  be a given vector in  $\mathbb{R}^3$  and  $A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  a transformation defined by

$$A(\vec{x}) = \vec{a} \times \vec{x} + \vec{x}.$$

Prove that the transformation  $A$  is linear.

3. Find the matrices of the following linear transformations:

- a) Rotation by 90 degrees counterclockwise.
- b) Rotation by 45 degrees counterclockwise.
- c) Reflection over the line  $y = x$ .
- d) Scaling given by  $(x, y) \mapsto (\alpha x, \beta y)$  for given scalars  $\alpha$  and  $\beta$ .

*Solution:*

- a)  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
- b)  $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$
- c)  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- d)  $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$

4. Let  $T$  be a linear transformation from the vector space  $\mathbb{R}^2$  to itself given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x + y \\ x + 3y \end{bmatrix}.$$

- Find the matrix  $A$  of  $T$  with respect to the standard basis  $\{i, j\}$ .
- Verify that the vectors  $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  form a basis of  $\mathbb{R}^2$ .
- Find the matrix  $B$  of  $T$  with respect to the basis  $\{v_1, v_2\}$ .
- Compute  $Bv_1$ . What do you observe?

*Solution:*

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

5. Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  and  $S: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be two linear transformations.

- Prove that the composition  $S \circ T$  is a linear transformation.

- Suppose that the matrix of  $T$  is  $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 3 \end{bmatrix}$  and the matrix of  $S$  is  $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ . Compute explicitly a formula for  $S(T(\begin{bmatrix} x \\ y \end{bmatrix}))$ .

- Find the matrix of the composition  $S \circ T$ .
- Compute the matrix product  $BA$ . What do you notice?

*Solution:*

- $S(T(\begin{bmatrix} x \\ y \end{bmatrix})) = \begin{bmatrix} 5y \\ y \end{bmatrix}$ .

- $\begin{bmatrix} 0 & 5 \\ 0 & 1 \end{bmatrix}$

- $BA = \begin{bmatrix} 0 & 5 \\ 0 & 1 \end{bmatrix}$

6. Let  $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ . Show that  $\det(A)$  equals the area of parallelogram spanned by the vectors  $A\vec{i}$  and  $A\vec{j}$ .