Polyà theory

Burnside lemma:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

$X^g = \{ x \in X ;\ g \overset{\text{delovanje}}{\cdot} x = x \}$

X - mn.

G - grupa

G deluje na X

Polya's theorem:

r colors

$$\#\text{ of colorings of pearls (elts of } X) = \frac{1}{|G|} \sum_{g \in G} r^{c(g)}$$

$c(g)$ = # cycles of g (štejemo tudi cikle dolžine 1)

① $|X| = n$, $|R| = r$ = # of colors

How many colorings if:

a) $G = \{id\}$

b) $G = S_n$?

a) $\frac{1}{1} r^n = r^n$

b) $\frac{1}{n!} \sum_{g \in S_n} r^{c(g)} = \frac{1}{n!} \sum_{k=1}^{n} c(n,k)\, r^k$

$$= \frac{1}{n!}\, r(r+1) \cdots (r+n-1) = \binom{r+n-1}{n}$$

|| weak compositions n + r členi:

$x_1 + \dots + x_r = n \qquad x_i \geq 0$

② # of colorings of faces of a tetrahedron with r colors?

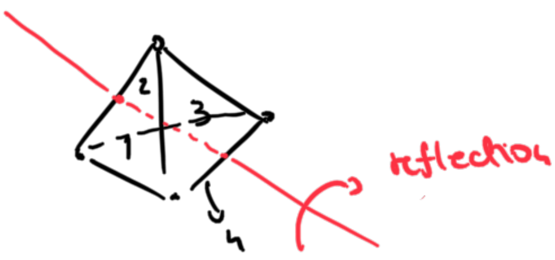
X = faces of a tetrahedron

G = symmetries of a —||—

$G = \{ id, \text{ rotations by } 120°, \text{ reflections} \}$

$4 \cdot 2 = 8$ possibilities

vertex, 2 rotations



3 possibilities

$$|G| = \underset{id}{1} + 8 + 3 = 12$$

$$\# \text{ of colorings} = \frac{1}{|G|} \sum_{g \in G} r^{c(g)}$$

$c(g)$:

$g = id \quad : \quad 4$

$g =$ rotation by $120°$: $\overset{\text{upr.}}{(4)\,(1\,2\,3)}$ 2 circle

di $(4)\,(1\,3\,2)$

$g =$ reflection : $\overset{\text{upr.}}{(12)\,(34)}$ 2 circle

$$\# \text{ of colorings} = \frac{1}{12}\left(r^4 + 8r^2 + 3r^2\right)$$
$$= \frac{1}{12}\left(r^4 + 11r^2\right)$$

Def. $g \in S_n$: $\alpha_i(g) = \#$ i-cycles of g

$$\left(\text{so} \quad \sum_i \alpha_i(g) = c(g) \right)$$

Cyclic index of a group $G \leq S_n$:

$$Z_G(t_1, \dots, t_n) = \frac{1}{|G|} \sum_{g \in G} t_1^{\alpha_1(g)} t_2^{\alpha_2(g)} \dots t_n^{\alpha_n(g)}$$

$n = |X|$

From lectures : Z_{C_n}, Z_{D_n}, Z_{cube} !!

of faces (or vertices)

Primer: $G =$ symmetries of a tetrahedron:

$$Z_G(t_1, \dots, t_4) = \frac{1}{12}\left(t_1^4 + 8\, t_1^1 \cdot t_3^1 + 3\, t_2^2\right)$$

$\#$ of colorings with r colors $= Z_G(r, \dots, r)$

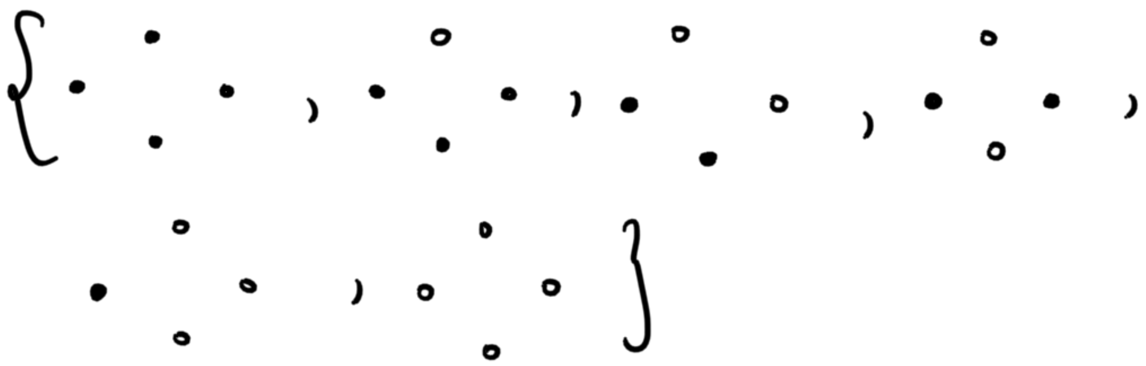
- $E \subseteq R^{\mathcal{X}}$ subset of colorings of the set $\mathcal{X}$
- Let $k_i(b) = |b^{-1}(i)|$ for $b \in E$

$$\underbrace{|b^{-1}(i)|}_{\#\text{ elts of color } i}$$

Define:

$$U_E(y_1, \dots, y_r) = \sum_{b \in E} y_1^{k_1(b)} \cdots y_r^{k_r(b)}$$

Example: $n = 4$ $r = 2$

Let $E =$ 

$\bullet$ ... index 1

$\circ$ ... index 2

$$\Rightarrow U_E(y_1, y_2) = y_1^4 + y_1^3 y_2 + 2 y_1^2 y_2^2 + y_2^4$$

How to count the $\#$ of colorings with specified number of elts of each color:

Theorem: $E \subseteq R^{\mathcal{X}}$, G group that acts of $\mathcal{X}$.
E .. such that it contains $\exists!$ element from each orbit.

$$U_E(y_1, \dots, y_r) = Z_G\left(y_1 + \dots + y_r,\ y_1^2 + \dots + y_r^2,\ \dots,\ y_1^n + \dots + y_r^n\right)$$

In particular: $U_E(1, 1, 1, \dots, 1) = Z_G(r, \dots, r)$
$= \#$ usual Polya theorem.

③ # of colorings of bracelets with 4 pearls,
2 black, 1 green, 1 red ?
① ② ③

# of colorings with a_i elts of color i is

$[y_1^{a_1} \cdots y_r^{a_r}]\, U_E(y_1, \ldots, y_r)$

$\cdot\ Z_G$? $G = D_4$ — 4 + 4 (rot, rot + refl.)

Dihedral group with 8 elts

From lectures:

$$Z_{D_n}(t_1, \ldots, t_n) = \frac{1}{2n}\left(\sum_{d \mid n} \Phi(d)\, t_d^{n/d} + \begin{cases} n\, t_1 t_2^{\frac{n-1}{2}} ; & n \text{ odd} \\ \left(t_2^{\frac{n}{2}} + t_1^2 t_2^{\frac{n-2}{2}}\right)\frac{n}{2} ; & n \text{ even} \end{cases}\right)$$

$$Z_{D_4}(t_1, \ldots, t_4) = \frac{1}{8}\Big(\underbrace{\Phi(1)}_{=1} t_1^4 + \underbrace{\Phi(2)}_{=1} t_2^2 + \underbrace{\Phi(4)}_{=2} t_4^1 + 2t_2^2 + 2t_1^2 t_2\Big)$$

$$= \frac{1}{8}\left(t_1^4 + \underline{t_2^2} + 2t_4 + \underline{2t_2^2} + 2t_1^2 t_2\right)$$

$$U_E(y_1, y_2, y_3) = Z_{D_4}(y_1 + y_2 + y_3,\ y_1^2 + y_2^2 + y_3^2,\ y_1^3 + y_2^3 + y_3^3,\ y_1^4 + y_2^4 + y_3^4)$$

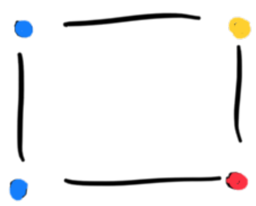
$$= \frac{1}{8}\Big((y_1 + y_2 + y_3)^4 + 3(y_1^2 + y_2^2 + y_3^2)^2 + 2(y_1^4 + y_2^4 + y_3^4) + 2(y_1 + y_2 + y_3)^2(y_1^2 + y_2^2 + y_3^2)\Big)$$

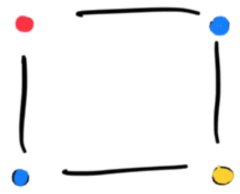
colorings with 2 pearls of color 1,
1 pearl of color 2,
1 pearl of color 3

$$= [y_1^2 y_2 y_3]\, U_E(y_1, y_2, y_3) = \frac{1}{8}\left[\binom{4}{2,1,1} + 0 + 0 + 2 \cdot 2\right] = 2$$

Recall :

$$(y_1 + y_2 + y_3)^n = \sum_{i,j,k} \binom{n}{i,j,k} y_1^i y_2^j y_3^k$$





indeed

(4) # of colorings of faces of a cube with 3 green, 2 blues & 1 red faces.

$$Z_{cube}(t_1, t_2, t_3, t_4, t_5, t_6) =$$

$$= \frac{1}{24}\left(t_1^6 + 6\, t_1^2 t_4 + 3\, t_1^2 t_2^2 + 8\, t_3^2 + 6\, t_2^3 \right)$$

what you are coloring

X = faces of a cube

G = symmetries of faces of a cube

$$Z_{cube}(y_1 + y_2 + y_3, \ldots, y_1^6 + y_2^6 + y_3^6) =$$

$$= \frac{1}{24}\Big[(y_1 + y_2 + y_3)^6 + 6\,(y_1 + y_2 + y_3)^2 (y_1^4 + y_2^4 + y_3^4) + 3\,(y_1 + y_2 + y_3)^2 (y_1^2 + y_2^2 + y_3^2)^2 + 8\,(y_1^3 + y_2^3 + y_3^3)^2 + 6\,(y_1^2 + y_2^2 + y_3^2)^3 \Big]$$

$$[y_1^3 y_2^2 y_3] \cdots = \frac{1}{24}\left[\binom{6}{3,2,1} + 0 + 3 \cdot 2 \cdot 2 + 0 + 0 \right]$$