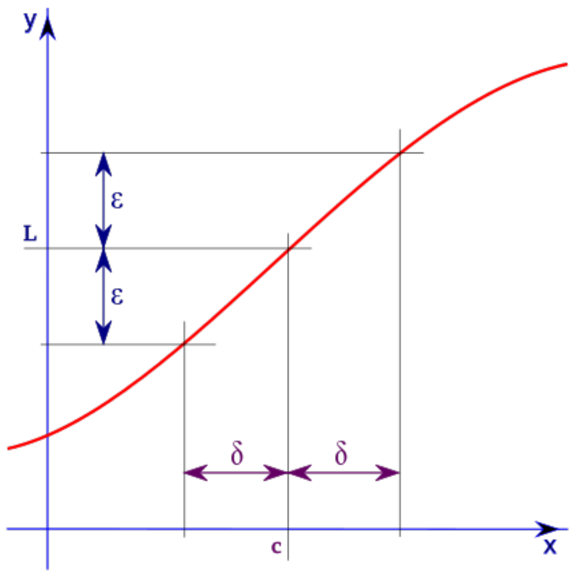
Continuity

1. Suppose we have a function $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}$. Explain in words the (ε, δ)-definition of continuity at a point $c \in A$. You might find the diagram below helpful.



   Give an example of a discontinuous function.

2. Discuss the continuity of the following functions.
   (a) $x^2 + x + 2$
   (b) The sign function $\text{sgn}(x)$.
   (c) The floor function $\lfloor x \rfloor$.
   (d) The function $\sin\left(\frac{1}{x}\right)$ on $(0, \infty)$.
   (e) $x - \lfloor x \rfloor$.
   (f) $\sin(2\pi(x - \lfloor x \rfloor))$.
   (g) $f(z) = \inf_{n \in \mathbb{N}} z^n$.

3. Find out whether the given function is continuous or discontinuous :

   1. $f(x) = \begin{cases} \frac{x^2-9}{x+3} & : \; x \neq -3 \\ 6 & : \; x = -3 \end{cases}$

2. $f(x) = \begin{cases} 3 + x^2 & : x \leq 0 \\ \frac{\sin(3x)}{x} & : x > 0 \end{cases}$

3. $f(x) = \begin{cases} \arctan x & : x \in (-\infty, 0] \\ x^3 & : x \in (0, 1] \\ 2 - x & : x \in (1, \infty) \end{cases}$

4. $f(x) = \frac{1}{e^x - 1}$

5. $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

6. $f(x) = \begin{cases} \frac{2x-8}{\sqrt{x-1}-\sqrt{3}} & : x \neq 4 \\ 2 & : x = 4 \end{cases}$

4. (*Intermediate value theorem.*) Suppose that $f\colon [a, b] \to \mathbb{R}$ is a continuous function on a closed, bounded interval. Then for every d strictly between $f(a)$ and $f(b)$ there is a point $a < c < b$ such that $f(c) = d$.

   We give some examples to show that all of the hypotheses in this theorem are necessary.

   a) Define $f\colon [-1, 1] \to \mathbb{R}$ by

   $$f(x) = \begin{cases} -1 & : \text{ if } -1 \leq x < 0 \\ 1 & : \text{ if } 0 \leq x \leq 1 \end{cases}.$$

   Then $f(-1) < 0$ and $f(1) > 0$, but f does not vanish at any point in its domain. Why does the theorem fail?

   b) Let $K = [-2, -1] \cup [1, 2]$ and define $f\colon K \to \mathbb{R}$ by

   $$f(x) = \begin{cases} -1 & : \text{ if } -2 \leq x \leq -1 \\ 1 & : \text{ if } 1 \leq x \leq 2 \end{cases}.$$

   Then $f(-2) < 0$ and $f(2) > 0$, but f does not vanish at any point in its domain. Why does the theorem fail?

5. Show that the function $f\colon [1, 2] \to \mathbb{R}$ given by $f(x) = x^2 - 2$ has exactly one root.

6. (*Extreme value theorem.*) Suppose that $f\colon [a, b] \to \mathbb{R}$ is a continuous function on a closed, bounded interval. Then $f([a, b]) = [m, M]$ is a closed, bounded interval.

   Define $f\colon [-1, 1] \to \mathbb{R}$ by $f(x) = x - x^3$. Find M such that

   $$f([-1, 1]) = [-M, M].$$