

Axioms of the real numbers

1. In class we have seen the least upper bound of a set $S \subseteq \mathbb{R}$, also known as the *supremum* of S .

- a) What is the analogous concept for lower bounds?
- b) What does the completeness axiom tell us about this concept?

2. Find the suprema and infima of the following sets. Is any of them attained?

- a) $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$
- b) $\left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\}$
- c) $\left\{ \frac{1}{3n^2 + 5} \mid n \in \mathbb{N} \right\}$
- d) $\left\{ \frac{n}{m} \mid m, n \in \mathbb{N}, m + n \leq 10 \right\}$

3. For these, we only need the field axioms of \mathbb{R} .

- a) Show that the additive identity 0 is unique.
- b) Prove that $0 \cdot x = 0$ for every $x \in \mathbb{R}$.
- c) Recall that $-x$ denotes the additive inverse of $x \in \mathbb{R}$. Prove that $(-1)x = -x$ for all $x \in \mathbb{R}$.
- d) Show that the additive inverse is unique.

For the next one, we also need the order axioms:

- e) Show that $1 > 0$; that is, $0 \leq 1$ and $1 \neq 0$.

4. An ordered field F has the Archimedean property if, given any positive x and y in F there is an integer $n > 0$ so that $nx > y$. Show that this property follows from the completeness axiom.