NAZADNJE SMO OBRAVNAVALI t.i. <u>FUBINIJEV IZREK</u>:

ČE SO VSE FUNKCIJE INTEGRABILNE NA USTREZNIH OBMOČJIH, VELJA:

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

TA IZREK POVE, KAKO DVOJNE INTEGRALE NA PRAVOKOTNIKIH IZRAČUNAMO V PRAKSI S PREVEDBO NA DVA DOLOČENA INTEGRALA ENE SPREMENLJIVKE.

DOKAZ:

$$g(x) = \int_c^d f(x,y)\,dy \quad \text{INTEGRAL S PARAMETROM}$$

DOKAŽIMO, DA JE

$$\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b g(x)\,dx$$

RIEMANNOVA VSOTA ZA DESNI INTEGRAL IMA OBLIKO:

$$R(g) = \sum_{j=1}^{n} g(\xi_j)(x_j - x_{j-1})$$

ZA $a = x_0 < x_1 < \dots < x_n = b$ IN $\xi_j \in [x_{j-1}, x_j]$.

UPOŠTEVAMO DEFINICIJO FUNKCIJE g IN DOBIMO:

$$R(g) = \sum_{j=1}^{n} \int_c^d f(\xi_j, y)\,dy\,(x_j - x_{j-1})$$

KLJUČNI TRIK JE, DA SEDAJ INTERVAL $[c,d]$ RAZBIJEMO NA INTERVALE $[y_{k-1}, y_k]$ IN $\int_c^d$ ZAPIŠEMO KOT:

$$\int_c^d f(\xi_j, y)\,dy = \sum_{k=1}^{m} \int_{y_{k-1}}^{y_k} f(\xi_j, y)\,dy$$

R-VSOTA ZA $\int_{y_{k-1}}^{y_k}$

TOREJ IMAMO:

$$R(g) = \sum_{j,k=1}^{n,m} \int_{y_{k-1}}^{y_k} f(\xi_j, y)\,dy\,(x_j - x_{j-1}) = \sum_{j,k=1}^{n,m} f(\xi_j, \eta_k)(x_j - x_{j-1})(y_k - y_{k-1})$$

$$= \sum_{j,k=1}^{n,m} f(\xi_j, \eta_k)\, P_{j,k} \quad \text{ZA } \eta_k \in [y_{k-1}, y_k]$$

SPOMNIMO:

$$S(f,D) = \sum_{j,k=1}^{n,m} M_{jk}\, P_{jk} \qquad s(f,D) = \sum_{j,k=1}^{n,m} m_{jk}\, P_{jk}$$

ZA M_{jk} IN m_{jk} SUPREMUM OZ INFIMUM f NA P_{jk}.
ZGORAJ SMO DOKAZALI, DA VELJA

$$s(f,D) \leq R(g) \leq S(f,D)$$

KER STA f IN g INTEGRABILNI, V LIMITI DOBIMO ENAKOST.

PRIMER: $f = x + 2y$ NA $[0,1] \times [0,1]$

$$I = \int_0^1 \left(\int_0^1 (x+2y)\, dy \right) dx = \int_0^1 xy + y^2 \Big|_{y=0}^{y=1} dx$$

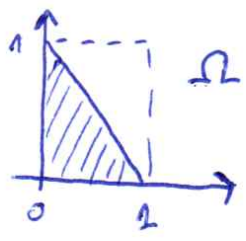
$$= \int_0^1 (x+1)\, dx = \frac{x^2}{2} + x \Big|_0^1 = \frac{1}{2} + 1 = \frac{3}{2}$$

DRUG VRSTNI RED:

$$I = \int_0^1 \left(\int_0^1 (x+2y)\, dx \right) dy = \int_0^1 \frac{x^2}{2} + 2xy \Big|_{x=0}^{x=1} dy =$$

$$= \int_0^1 \left(\frac{1}{2} + 2y \right) dy = \frac{y}{2} + y^2 \Big|_{y=0}^{y=1} = \frac{3}{2}$$

NOVO VPRAŠANJE: KAJ ČE f INTEGRIRAMO PO OBMOČJU, KI NI PRAVOKOTNIK. NA PRIMER PO TRIKOTNIKU Ω S KRAJIŠČI V $(0,0), (1,0), (0,1)$.



FORMALNA DEFINICIJA:

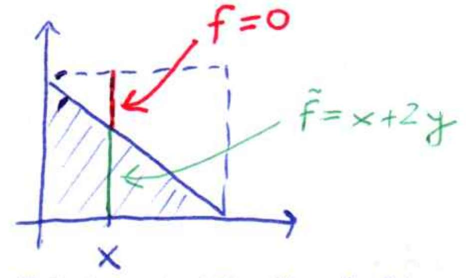
$$\tilde{f}(x,y) = \begin{cases} x + 2y \;; & (x,y) \in \Omega \\ 0 \;; & (x,y) \in [0,1] \times [0,1] \setminus \Omega \end{cases}$$

$$I = \iint_{[0,1] \times [0,1]} \tilde{f}(x,y)\, dx\, dy$$

KAJ TO POMENI V PRAKSI?

$$I \overset{\text{FUBINI}}{=} \int_0^1 \Big(\underbrace{\int_0^1 \tilde{f}(x,y)\, dy}_{} \Big)\, dx$$

PRI FIKSNEM
$x \in [0,1]$ GLEDAMO
$\tilde{f}(x,y)$



STRANICA JE: $y = 1-x$

SKLEP:

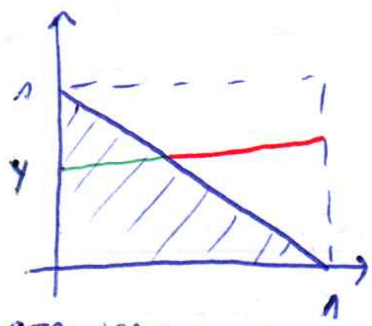
$$I = \int_0^1 \Big(\int_0^{1-x} (x+2y)\, dy \Big) dx = \int_0^1 (xy+y^2)\Big|_{y=0}^{y=1-x} dx =$$

$$= \int_0^1 x(1-x) + (1-x)^2 \, dx = \int_0^1 \underbrace{x - \cancel{x^2} + 1 - 2x + \cancel{x^2}}_{1-x} dx = 1 - \frac{1}{2} = \frac{1}{2}$$

KAJ SE ZGODI, ČE ZAMENJAMO SMER OZ. VRSTNI RED?

$$I \overset{\text{FUBINI}}{=} \int_0^1 \Big(\underbrace{\int_0^1 \tilde{f}(x,y)\, dx}_{} \Big) dy$$

PRI FIKSNEM
$y \in [0,1]$ GLEDAMO
$\tilde{f}(x,y)$



STRANICA: $y = 1-x$
$x = 1-y$

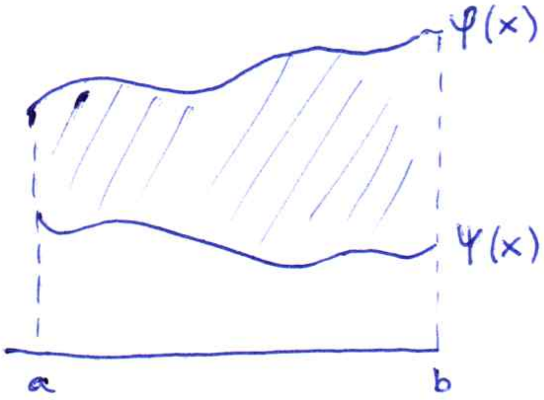
SKLEP:

$$I = \int_0^1 \Big(\int_0^{1-y} (x+2y)\, dx \Big) dy = \int_0^1 \Big(\frac{x^2}{2} + 2xy\Big) \Big|_0^{1-y} dy =$$

$$= \int_0^1 \Big(\frac{(1-y)^2}{2} + 2y - 2y^2\Big) dy = -\frac{(1-y)^3}{6} + y^2 - \frac{2y^3}{3} \Big|_0^1 =$$

$$= 1 - \frac{2}{3} + \frac{1}{6} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$$

KAJ VELJA V SPLOŠNEM?



$$\Omega = \{(x,y) : x \in [a,b],\ \varphi(x) \geq y \geq \psi(x)\}$$

φ IN ψ ZVEZNI NA $[a,b]$

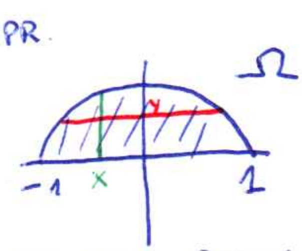
F INTEGRABILNA

$$I = \iint_{\Omega} F(x,y)\,dx\,dy = \int_a^b \left(\int_{\psi(x)}^{\varphi(x)} F(x,y)\,dy \right) dx$$

OPOMBE:

(1) IZ PRIMERA NA TRIKOTNIKU MORDA IZGLEDA, DA JE VLOGA x IN y V INTEGRALU SIMETRIČNA, A TEMU V SPLOŠNEM NI TAKO.

NPR.



POLOVICA DISKA

$x^2+y^2=1$

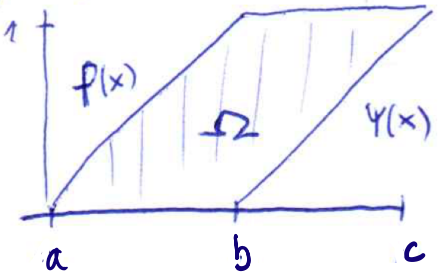
$\boxed{y = \pm\sqrt{1-x^2}}$

$\boxed{x = \pm\sqrt{1-y^2}}$

$$I = \int_{-1}^{1} \left(\int_0^{\sqrt{1-x^2}} F(x,y)\,dy \right) dx =$$

$$= \int_0^1 \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} F(x,y)\,dx \right) dy$$

(2) VČASIH JE TREBA OBMOČJE RAZBITI NA DVA DELA. (ALI TUDI VEČ DELOV).



$$\iint_{\Omega} F(x,y)\,dx\,dy = \int_a^b \left(\int_0^{\varphi(x)} F(x,y)\,dy \right) dx + \int_b^c \left(\int_{\psi(x)}^{1} F(x,y)\,dy \right) dx$$

(3) DO SLEJ SMO PISALI OKLEPAJE, DA SMO OPOZARJALI NA USTREZEN VRSTNI RED. ODSLEJ PIŠEMO.

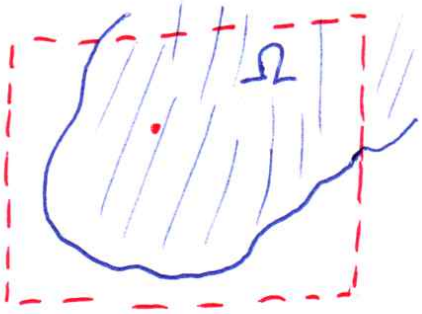
$\iint_{\Omega} F(x,y)\,dx\,dy$ SPLOŠNI INTEGRAL, BREZ KONKRETNIH MEJ.

$\int_a^b \int_{\psi(x)}^{\psi(x)} F(x,y)\,dy\,dx$ ALI $\int_a^b dx \int_{\varphi(x)}^{\psi(x)} F(x,y)\,dy$ NAJPREJ PO y, NATO PO x.

POSPLOŠENI DVOJNI INTEGRAL

DOSLEJ SMO OBRAVNAVALI LE OMEJENA OBMOČJA $\Omega \subseteq \mathbb{R}^2$ IN OMEJENE FUNKCIJE f. SEDAJ SI OGLEJMO POSPLOŠITEV INTEGRALA TUDI ZA NEOMEJEN PRIMER.

TIP 1: NEOMEJENO OBMOČJE



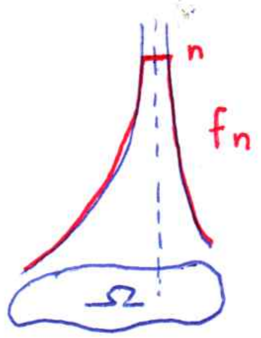
$K_n = [-n, n] \times [-n, n]$

DEFINIRAMO:

$$\tilde{f} = \begin{cases} f; & (x,y) \in \Omega \\ 0; & \mathbb{R}^2 \setminus \Omega \end{cases}$$

$$\iint_\Omega f(x,y)\,dx\,dy = \lim_{n \to \infty} \iint_{K_n} \tilde{f}(x,y)\,dx\,dy$$

TIP 2: NEOMEJENA FUNKCIJA



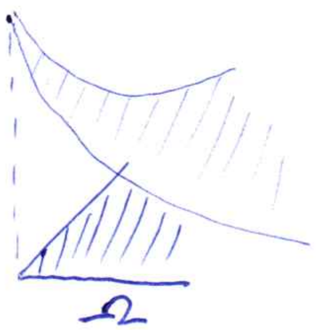
DEFINIRAMO:

$$f_n(x,y) = \begin{cases} f(x,y); & \text{ZA } |f(x,y)| \le n \\ 0; & \text{sicer} \end{cases}$$

$$\iint_\Omega f(x,y)\,dx\,dy = \lim_{n \to \infty} \iint_\Omega f_n(x,y)\,dx\,dy$$

V PRAKSI ZA IZRAČUN INTEGRALA OBEH TIPOV UPORABIMO KAR POSPLOŠENI FUBINIJEV IZREK, KI POVE ZGOLJ, DA SE NAMESTO MEJA RAČUNA LIMITE.

PRIMER: $f(x,y) = e^{-(x+y)}$ NA $\Omega = [0, \infty) \times [0, \infty)$



$$\iint_\Omega e^{-(x+y)}\,dx\,dy = \int_0^\infty \int_0^\infty e^{-(x+y)}\,dy\,dx =$$

$$= \int_0^\infty -e^{-(x+y)}\Big|_{y=0}^{y=\infty}\,dx = \int_0^\infty 0 + e^{-x}\,dx$$

$$= -e^{-x}\Big|_{x=0}^{x=\infty} = 0 - 1 = \underline{\underline{1}}$$

ZAMENJAVA SPREMENLJIVK V DVOJNEM INT.

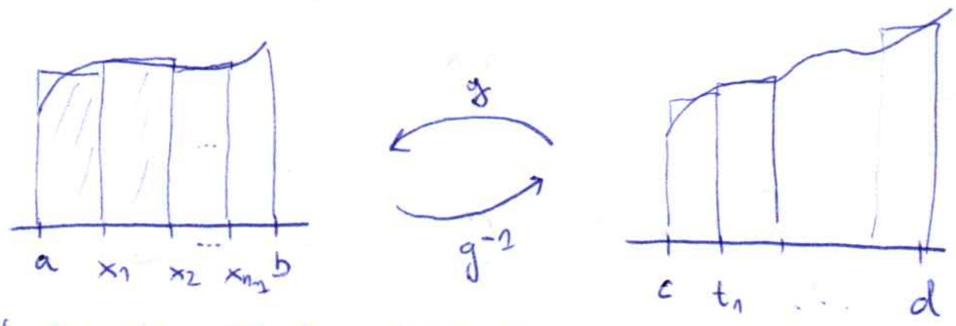
SPOMNIMO:

$$\int_a^b f(x)\,dx = \int_c^d f(g(t))\, g'(t)\, dt$$

$x = g(t)$ NOVA SPR. PODANA Z BIJ. FN. $g : [c,d] \to [a,b]$

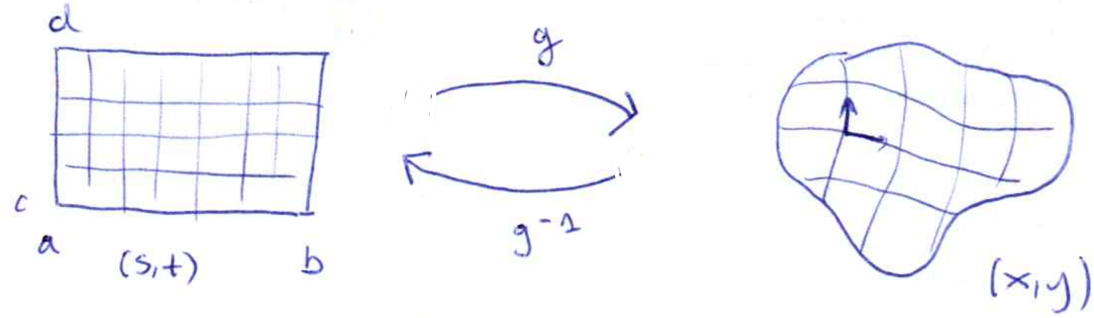
$dx = g'(t)\, dt$

KAKO SMO PRIŠLI DO TEGA PRAVILA?



V PRINCIPU GRE ZA ENAKO ŠTEVILO DELILNIH TOČK IN ISTO R-VSOTO, LE RAZPOREDITEV OZ. "HITROST SEŠTEVANJA" JE DRUGAČNA ZATO SE PODA $g'(t)$.

ANALOGIJA ZA FUNKCIJE DVEH SPR.:



PLOŠČINA "KVADRATKOV" NA DESNI, JE PRIBLIŽNO:

$$|\det J_g| \text{, KJER JE } J_g = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix}$$

IZREK:

NAJ BOSTA $\Omega, D \subset \mathbb{R}^2$ OBMOČJI IN $g : \Omega \to D$ RAZREDA $\mathcal{C}^1$.
ČE JE $f : D \to \mathbb{R}^2$ ZV. + $\det J_g \neq 0$ NA Ω, VELJA:

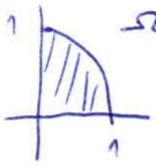
$$\iint_D f(x,y)\,dx\,dy = \iint_\Omega f(x(s,t), y(s,t))\, |\det J_g(s,t)|\, ds\,dt$$

PRIMER: POLARNE KOORDINATE

$$x = r\cos\varphi \qquad x_r = \cos\varphi \qquad x_\varphi = -r\sin\varphi \qquad J = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{bmatrix}$$

$$y = r\sin\varphi \qquad y_r = \sin\varphi \qquad y_\varphi = r\cos\varphi \qquad \det J = r$$

IZRAČUNAJMO:



$$I = \iint_\Omega x\,dx\,dy$$

NA OBIČAJEN NAČIN

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} x\,dy = \int_0^1 x\sqrt{1-x^2}\,dx = \frac{1}{2}\int_0^1 \sqrt{t}\,dt = \frac{1}{3}$$

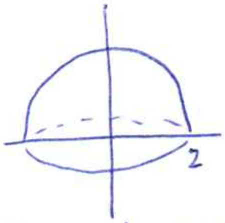
$$t = 1 - x^2$$

V POLARNIH KOORD:

$$\int_0^1 dr \int_0^{\frac{\pi}{2}} \underbrace{r\cos\varphi}_{x(r,\varphi)} \cdot \underbrace{r}_{|\det J|}\,d\varphi = \int_0^1 r^2 \sin\varphi \Big|_0^{\frac{\pi}{2}} dr = \frac{r^3}{3}\Big|_0^1 = \frac{1}{3}$$

JACOBI

PRIMER: IZRAČUNAJMO VOLUMEN VRTENINE ZA $y = 4 - x^2$, $x \in [0,2]$ OKOLI OSI Z.



$F(x,y) = 4 - x^2 - y^2$

$$V = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)\,dy = \text{TEŽAVNO.}$$

KARTEZIČNE

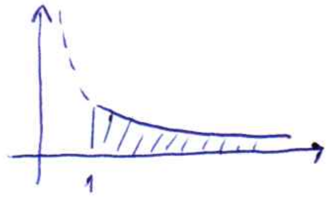
$$V = \int_0^{2\pi} d\varphi \int_0^2 (4 - r^2) \cdot r\,dr = 2\pi\left(2r^2 - \frac{r^4}{4}\right)\Big|_0^2 = 8\pi$$

POLARNE

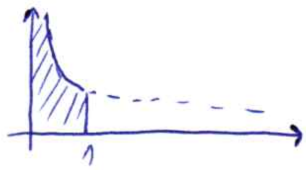
JACOBI

PRIMER: PRI POSPLOŠENIH INTEGRALIH FUNKCIJE 1 SPR. VELJA NASLEDNJE.

$$\int_1^\infty \frac{dx}{x^\alpha} \quad \text{OBSTAJA ZA } \alpha > 1.$$



$$\int_0^1 \frac{dx}{x^\alpha} \quad \text{OBSTAJA ZA } \alpha < 1.$$

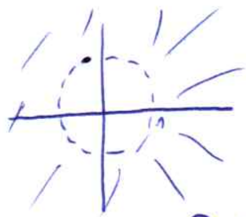


t.j. ZA KONVERGENCO V ∞ MORA BITI FUNKCIJA V NIČ HITREJE KOT $1/x$, ZA KONVERGENCO V 0 PA MORA BITI STOPNJA POLA STROGO MANJŠA OD PRVE.

OGLEJMO SI, KAJ SE ZGODI NA $\mathbb{R}^2$ OZ. KO TO ISTO FUNKCIJO ZAROTIRAMO OKOLI OSI y.

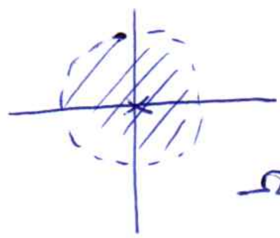
$$f(x,y) = \frac{1}{\left(\sqrt{x^2+y^2}\right)^\alpha} = \frac{1}{r^\alpha} \quad \text{V POLARNIH KOORDINATAH.}$$

INTEGRIRAMO PO:



Ω_1

DA PREVERIMO DOGAJANJE V ∞



Ω_2

DA PREVERIMO POL

$$\iint_{\Omega_1} f\,dx\,dy = \int_0^{2\pi} d\varphi \int_1^\infty \frac{1}{r^\alpha} \cdot \underbrace{r}_{\text{JACOBI}}\, dr = \underbrace{2\pi \int_1^\infty \frac{1}{r^{\alpha-1}}\, dr}_{\text{OBSTAJA ZA } \alpha-1>1}$$

$$\boxed{\alpha > 2}$$

$$\iint_{\Omega_2} f\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^1 \frac{1}{r^\alpha} \cdot r\, dr = \underbrace{2\pi \int_0^1 \frac{1}{r^{\alpha-1}}\, dr}_{\text{OBSTAJA ZA } \boxed{\alpha<2}}$$

SKLEP: MEJA JE PRI POLU DRUGE STOPNJE.