

1. kolokvij iz Teorije mere (P)

6. december 2010

(1) Calculate the limit

$$\lim_{n \rightarrow \infty} \int_{\frac{1}{n}}^1 \frac{n+x}{n(x^2+2)} \sin\left(\frac{n\pi-2x}{2n}\right) dx.$$

(2) Let $(X, \mathcal{A}, \mu)$ be a measureable space. For an arbitrary set $E \in \mathcal{A}$ we define

$$\mu_0(E) = \sup\{\mu(F) : \mu(F) < \infty, F \in \mathcal{A} \text{ and } F \subseteq E\}.$$

Prove that μ_0 is a positive measure on $(X, \mathcal{A})$.

(3) Let X and Y be topological spaces, and let $\mathcal{B}_X$ and $\mathcal{B}_Y$ Borel σ -algebras on X and Y respectively. Let $f : X \rightarrow Y$ be an open and injective mapping. Prove $f(\mathcal{B}_X) \subseteq \mathcal{B}_Y$.

(4) Let $(X, \mathcal{A}, \mu)$ be a measureable space and let $f : X \rightarrow \mathbb{R}$ be a nonnegative function such that $\int_X f d\mu < \infty$. Prove that for every $\epsilon > 0$ there exists a subset $E \in \mathcal{A}$ with finite measure such that

$$\int_E f d\mu > \int_X f d\mu - \epsilon.$$