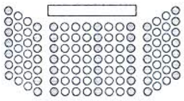
Matematika 1: 1. kolokvij (B)

2. 12. 2013

Čas pisanja je 90 minut. Možno je doseči 100 točk. Veliko uspeha!



Sedež (2.05)

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Ime in priimek

Vpisna številka

1. naloga

S pomočjo matematične indukcije dokaži naslednji trditvi.

a) Za vsako naravno število $n \in \mathbb{N}$ velja

$$3 + 9 + 15 + \cdots + (6n - 3) = 3n^2.$$

b) Število $5^{n-1} + 11^{2n+1}$ je deljivo 54 za vse $n \in \mathbb{N}$.

a) $n=1 \quad 3=3$ ✓

$n \to n+1$

$$3+9+15+\cdots+(6n-3)+(6(n+1)-3) =$$
$$= (3+9+15+\cdots+(6n-3)) + (6n+3) =$$
$$= 3n^2+6n+3 = 3(n^2+2n+1) = 3(n+1)^2$$

b) $n=1: \quad 1+11^3 = 1+1331 = 1332 = 4\cdot 9\cdot 37$ ✓

$$n \to n+1: \quad 5^n + 11^{2(n+1)+1} = 5^n + 11^{2n+3} = 5\cdot 5^{n-1} + 11^2\cdot 11^{2n+1}$$
$$= (4\cdot 5^{n-1} + 120\cdot 11^{2n+1}) + (5^{n-1} + 11^{2n+1}) =$$
$$= 4\cdot(5^{n-1} + 30\cdot 11^{2n+1}) + (5^{n-1} + 11^{2n+1})$$

deljivo s 4 po I. P.

2. naloga

Poišči vsa kompleksna števila $z \in \mathbb{C}$, ki rešijo enačbo

$$|z|^2 + iz^2 + i\overline{z^2} = 2i|z|^2 - 16(i-1)$$

in jih skiciraj v kompleksni ravnini.

$z = a + ib \Rightarrow$

$a^2 + b^2 + i(a^2 + 2abi - b^2) + i(a^2 - 2abi - b^2) = 2i(a^2+b^2) - 16(i-1)$

Re: $a^2 + b^2 - 2ab + 2ab = 16$

Im: $a^2 - b^2 + a^2 - b^2 = 2a^2 + 2b^2 - 16$

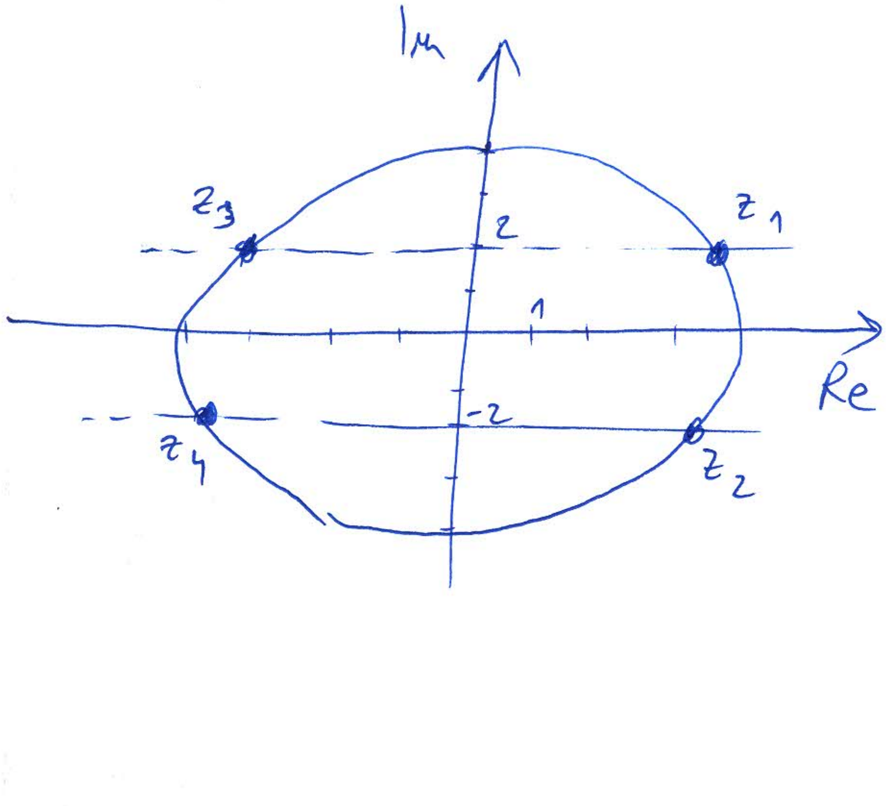
$\Rightarrow \quad a^2 + b^2 = 16$ in

$4b^2 = 16 \Rightarrow b^2 = 4 \Rightarrow b = \pm 2$

$\Rightarrow \quad a^2 = 16 - b^2 = 12 \Rightarrow a = \pm 2\sqrt{3}$

$\Rightarrow \quad z_1 = 2\sqrt{3} + 2i \qquad z_3 = -2\sqrt{3} + 2i$

$z_2 = 2\sqrt{3} - 2i \qquad z_4 = -2\sqrt{3} - 2i$



3. naloga

Funkcija f je podana s predpisom

$$f(x) = \ln\left(1 - e^{3x}\right).$$

a) Določi definicijsko območje funkcije f.

b) Dokaži, da je funkcija $f : \mathcal{D}_f \to \mathcal{Z}_f$ injektivna.

c) Določi inverzno funkcijo g k funkciji f.

d) Kaj je zaloga vrednosti funkcije f?

a) $D_f = \{x \in \mathbb{R} : 1 - e^{3x} > 0\} = \{x \in \mathbb{R} : e^{3x} < 1\}$

$= \{x \in \mathbb{R} : 3x < 0\} = (-\infty, 0)$

b) $\underline{f(x) = f(y) \Rightarrow x = y}$

$\ln(1 - e^{3x}) = \ln(1 - e^{3y}) \quad / e^{\cdot}$

$1 - e^{3x} = 1 - e^{3y} \Rightarrow e^{3x} = e^{3y} \Rightarrow 3x = 3y \Rightarrow x = y.$

c) $x = \ln(1 - e^{3y}) \Rightarrow e^{x} = 1 - e^{3y}$

$\Rightarrow e^{3y} = 1 - e^{x} \Rightarrow 3y = \ln(1 - e^{x})$

$\Rightarrow y = \frac{1}{3}\ln(1 - e^{x})$

$\Rightarrow g(x) = \frac{1}{3}\ln(1 - e^{x})$

d) $Z_f = \{f(x) : x \in (-\infty, 0)\} = (-\infty, 0)$

4. naloga

a) Izračunaj limito
$$\lim_{n\to\infty}(\sqrt{n^2+6n}-\sqrt{n^2+4n}).$$

b) Izračunaj limito
$$\lim_{n\to\infty}\left(\frac{n+4}{n-3}\right)^{\frac{n^3}{3n^2+1}}.$$

c) Zaporedje $\{a_n\}_{n\in\mathbb{N}}$ je podano z rekurzivno zvezo
$$a_{n+1}=\frac{1}{2}\sqrt{3a_n+1}$$
in prvim členom $a_1=3$. Dokaži, da je zaporedje $\{a_n\}_{n\in\mathbb{N}}$ konvergentno in izračunaj njegovo limito.

a) $$\lim_{n\to\infty}(\sqrt{n^2+6n}-\sqrt{n^2+4n}) = \lim_{n\to\infty}\frac{(n^2+6n)-(n^2+4n)}{\sqrt{n^2+6n}+\sqrt{n^2+4n}} =$$

$$= \lim_{n\to\infty}\frac{2n}{\sqrt{n^2+6n}+\sqrt{n^2+4n}} = \lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{6}{n}}+\sqrt{1+\frac{4}{n}}} = \frac{2}{1+1} = \underline{1}$$

b) $$\lim_{n\to\infty}\left(\frac{n+4}{n-3}\right)^{\frac{n^3}{3n^2+1}} = \lim_{n\to\infty}\left(1+\frac{7}{n-3}\right)^{\frac{n^3}{3n^2+1}} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n-3}{7}}\right)^{\frac{n^3}{3n^2+1}} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n-3}{7}}\right)^{\frac{n-3}{7}\cdot\frac{7}{n-3}\cdot\frac{n^3}{3n^2+1}} =$$

$$= e^{\lim_{n\to\infty}\frac{7n^3}{(n-3)(3n^2+1)}} = e^{\lim_{n\to\infty}\frac{7}{(1-\frac{3}{n})(3+\frac{1}{n^2})}} = \underline{\underline{e^{\frac{7}{3}}}}$$

c) $a_1=3,\quad a_2=\frac{1}{2}\sqrt{10}\leq 3$

$\{a_n\}_{n\in\mathbb{N}}$ pada

$a_{n+1}\leq a_n \quad \forall n\in\mathbb{N}$

$n=1: \quad a_2\leq a_1$ ✓

$$a_{n+2} - a_{n+1} = \frac{1}{2}\sqrt{3a_{n+1}+1} - \frac{1}{2}\sqrt{3a_n+1} =$$

$$= \frac{1}{2}\left(\sqrt{3a_{n+1}+1} - \sqrt{3a_n+1}\right) = \frac{1}{2} \cdot 3 \cdot \frac{a_{n+1}-a_n}{\sqrt{3a_{n+1}+1} + \sqrt{3a_n+1}}$$

Po I. P. $a_{n+1} \le a_n \Rightarrow a_{n+2} - a_{n+1} \le 0$

Zaporedje je navzdol omejeno z 0.

$\Rightarrow (a_n)_{n\in\mathbb{N}}$ konvergira

$$\Rightarrow \quad L = \frac{1}{2}\sqrt{3L+1} \Rightarrow 2L = \sqrt{3L+1}$$

$$\Rightarrow 4L^2 = 3L+1$$

$$\Rightarrow 4L^2 - 3L - 1 = 0$$

$$\Rightarrow (4L+1)(L-1) = 0$$

$\Rightarrow$ $L_1 = -\frac{1}{4}$ odpade, ker $a_n \geq 0 \ \forall n \in \mathbb{N}$. , $\boxed{L_2 = 1}$