MEASURE THEORY — EXAM 2

Time allowed: 120 min. Total no. of points: 100.
You may write in pen or pencil.

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$A \subset B \overset{\text{def}}{\Leftrightarrow} \forall x(x \in A \Rightarrow x \in B)$. 2^X je potenčna množica X. m je Lebesgueova mera na $\mathbb{R}$. $\mathcal{A}/\mathcal{B}$ je množica $\mathcal{A}/\mathcal{B}$-merljivih preslikav. $\mathcal{B}_X$ je Borelova σ-algebra na X pod standardno topologijo za X. $\overline{\mathcal{A}}^\mu$ je napolnitev σ-algebre $\mathcal{A}$ glede na mero μ na $\mathcal{A}$. Za merljiv prostor (X, Σ, μ) in za $p \in (0, \infty)$ je $\mathcal{L}^p(\mu) := \{f \in \Sigma/\mathcal{B}_\mathbb{C} : \int |f|^p d\mu < \infty\}$.

$A \subset B \overset{\text{def}}{\Leftrightarrow} \forall x(x \in A \Rightarrow x \in B)$. 2^X is the power set of X. m is the Lebesgue measure on $\mathbb{R}$. $\mathcal{A}/\mathcal{B}$ is the set of $\mathcal{A}/\mathcal{B}$-measurable maps. $\mathcal{B}_X$ is the Borel σ-algebra on X for the standard topology on X. $\overline{\mathcal{A}}^\mu$ is the completion of a σ-field $\mathcal{A}$ with respect to a measure μ on $\mathcal{A}$. For a measurable space (X, Σ, μ) and for $p \in (0, \infty)$, $\mathcal{L}^p(\mu) := \{f \in \Sigma/\mathcal{B}_\mathbb{C} : \int |f|^p d\mu < \infty\}$.

1. (25 points) Naj bo $(X, \mathcal{F}, \mu)$ prostor z mero. Dokaži, da je μ σ-končna, natanko tedaj ko je $(\mathcal{F}/\mathcal{B}_{(0,\infty]}) \cap \mathcal{L}^1(\mu) \neq \emptyset$. *Let $(X, \mathcal{F}, \mu)$ be a measure space. Prove that μ is σ-finite if and only if $(\mathcal{F}/\mathcal{B}_{(0,\infty]}) \cap \mathcal{L}^1(\mu) \neq \emptyset$.*

Solution: The condition is sufficient: if f is from the intersection that is being assumed non-empty then $X = \cup_{n\in\mathbb{N}}\{f \geq 1/n\}$. The condition is necessary. Let $X = \cup_{n\in\mathbb{N}} A_n$ with $A_n \in \mathcal{F}$, $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$, and where we may assume without loss of generality that $A_i \cap A_j = \emptyset$ for natural $i \neq j$. Take $f = \sum_{n\in\mathbb{N}} \frac{1}{n^2(\mu(A_n)\vee 1)} \mathbb{1}_{A_n}$.

2. Dana sta množica Ω in neka particija $(R_\lambda)_{\lambda\in\Lambda}$ množice Ω (torej $\Omega \supset R_\lambda \neq \emptyset$ za vse $\lambda \in \Lambda$; $\cup_{\lambda\in\Lambda} R_\lambda = \Omega$; $R_\lambda \cap R_\mu = \emptyset$ za $\lambda \neq \mu$ iz Λ). *There are given a set Ω and a partition $(R_\lambda)_{\lambda\in\Lambda}$ of the set Ω (so $\Omega \supset R_\lambda \neq \emptyset$ for all $\lambda \in \Lambda$; $\cup_{\lambda\in\Lambda} R_\lambda = \Omega$; $R_\lambda \cap R_\mu = \emptyset$ for $\lambda \neq \mu$ from Λ).*
   (a) (10 points) Določi $\sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\})$. *Determine $\sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\})$.*
   (b) (10 points) Dokaži, da je $f \in \sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\})/\mathcal{B}_{[-\infty,\infty]}$, če in samo če ima f števno zalogo vrednosti in je $f^{-1}(\{c\}) \in \sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\})$ za vse c iz te števne zaloge vrednosti. *Prove that $f \in \sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\})/\mathcal{B}_{[-\infty,\infty]}$, if and only if $f : \Omega \to [-\infty, \infty]$ has a denumerable range and $f^{-1}(\{c\}) \in \sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\})$ for all c from this denumerable range.*
   (c) (5 points) Ko je $\Lambda = \Omega$ in $R_\lambda = \{\lambda\}$ za $\lambda \in \Lambda$, kdaj (natanko) je $\sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\}) = 2^\Omega$? *When $\Lambda = \Omega$ and $R_\lambda = \{\lambda\}$ for $\lambda \in \Lambda$, when (precisely) is $\sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\}) = 2^\Omega$?*

Solution: Iskana σ-algebra je enaka

$$\{\cup_{\gamma\in\Gamma} R_\gamma : \Gamma \subset \Lambda \text{ števna ali s števnim komplementom v } \Lambda\}.$$

Dokaz je analogen dokazu za števno-koštevno σ-algebro. V posebnem, če je Λ števna (končna) je iskana σ-algebra kar enaka vsem možnim unijam elementov dane particije.

For the second part, the sufficiency of the conditions is essentially clear (owing to the facts that (i) σ-fields are closed under denumerable unions, (ii) the preimage of a set C under f is the same as the preimage of $C \cap \mathcal{R}f$, which is the union of the singletons of its members, and (iii) preimages commute with unions). Necessity; let $f \in \sigma_\Omega(\{R_\lambda : \lambda \in \Lambda\})/\mathcal{B}_{[-\infty,\infty]}$. Since singletons belong to $\mathcal{B}_{[-\infty,\infty]}$ we need only argue that the range of f is denumerable. For sure f must be constant on each R_λ, $\lambda \in \Lambda$. Then one verifies that f is constant on a set whose complement is the denumerable union of R_λs. The latter is true for indicators of measurable sets and so for a general function by the usual approximation argument.

Končno v primeru, ko particija sestoji iz singletonov Ω, je σ-algebra generirana z njo torej enaka 2^Ω natanko tedaj, ko je Ω števna. Namreč, neštevna množica ima podmnožico, ki ni niti števna, niti ni njen komplement števen.

Remark. If one endows the generated σ-field with the $0-1$ measure, μ, that is zero on sets that are countable unions of members of the partition, 1 otherwise, one obtains that $\int f d\mu$ is always well-defined, zero when Λ is denumerable, and otherwise is equal to the unique $x_0 \in [-\infty,\infty]$ for which $f^{-1}(\{x_0\})$ is a union of non-denumerably many members of the given partition.

3. Naj bo (X,Σ,μ) prostor z mero, $f = (f_n)_{n\in\mathbb{N}_0}$ zaporedje v $\Sigma/\mathcal{B}_\mathbb{R}$, in denimo, da je $f_0 = \lim_{n\to\infty} f_n$ v μ-meri. Dokaži: *Let (X,Σ,μ) be a measure space, $f = (f_n)_{n\in\mathbb{N}_0}$ a sequence in $\Sigma/\mathcal{B}_\mathbb{R}$, and suppose $f_0 = \lim_{n\to\infty} f_n$ in μ-measure. Prove:*
   (a) (13 points) Če obstaja $g \in \mathcal{L}^1(\mu)$, da je $|f_n| \leq g$ s.p.-μ za $n \in \mathbb{N}$, potem je $f_0 \in \mathcal{L}^1(\mu)$ in $\int f_0 d\mu = \lim_{n\to\infty} \int f_n d\mu$. *If there exists $g \in \mathcal{L}^1(\mu)$, such that $|f_n| \leq g$ a.e.-μ for $n \in \mathbb{N}$, then $f_0 \in \mathcal{L}^1(\mu)$ and $\int f_0 d\mu = \lim_{n\to\infty} \int f_n d\mu$.*
   (b) (12 points) Če je $f_n \geq 0$ s.p.-μ za $n \in \mathbb{N}$, potem je $f_0 \geq 0$ s.p.-μ in $\int f_0 d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$. *If $f_n \geq 0$ a.e.-μ for $n \in \mathbb{N}$, then $f_0 \geq 0$ a.e.-μ and $\int f_0 d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$.*

   Spomnimo (povejmo): Če $g_n \to g_0$ v μ-meri, potem obstaja podzaporedje g_{n_k}, ki konvergira k g_0 s.p.-μ. ***Recall*** *(be advised that): If $g_n \to g_0$ in μ-measure, then there exists a subsequence g_{n_k} that converges to g_0 a.e.-μ.*

Solution: The first part follows immediately from Lebesgue's dominated convergence theorem, the elementary observation that for a sequence of real number $(r_n)_{n\in\mathbb{N}_0}$, $\lim_{n\to\infty} r_n = r_0$ iff every subsequence of r admits a further subsubsequences on which this convergence holds true, and the given fact concerning the a.e. convergence of subsequences (noting that every subsequence of f converges to f_0 in μ-measure also).

The second part follows in a similar manner from Fatou's lemma, once it has been noted that for a sequence $(r_n)_{n\in\mathbb{N}_0}$ in $[-\infty,\infty]$, $r_0 \leq \liminf_{n\to\infty} r_n$ iff every subsequence of r admits a further subsubsequences on which this inequality holds true.

4. (25 points) Naj bo $(X,\mathcal{F},\mu)$ verjetnostni prostor, $X \in \mathcal{F}/\mathcal{B}_{(0,\infty)}$, $\alpha \in (0,1]$, $\phi : (0,\infty) \to \mathbb{R}$ konveksna z $\lim_{0+} \phi = 0$. Predpostavi, da je $\int X d\mu < \infty$. Dokaži, da je za $\alpha \in (0,1]$

(i.) $\phi(\alpha x) \leq \alpha\phi(x)$ za $x \in (0,\infty)$, in (ii.) $\int \phi \circ X d\mu \geq \alpha^{-1}\phi(\alpha \int X d\mu)$. *Let* $(X, \mathcal{F}, \mu)$ *be a probability space,* $X \in \mathcal{F}/\mathcal{B}_{(0,\infty)}$, $\alpha \in (0,1]$, $\phi : (0,\infty) \to \mathbb{R}$ *convex with* $\lim_{0+} \phi = 0$. *Assume* $\int X d\mu < \infty$. *Show that for* $\alpha \in (0,1]$ *(i.)* $\phi(\alpha x) \leq \alpha\phi(x)$ *for* $x \in (0,\infty)$, *and (ii.)* $\int \phi \circ X d\mu \geq \alpha^{-1}\phi(\alpha \int X d\mu)$.

Solution: (i.) For $\alpha = 1$ it is clear; otherwise let $\epsilon \in (0, x)$, then $\phi(\alpha x) = \phi(\alpha(x - \epsilon) + (1-\alpha)\frac{\alpha\epsilon}{1-\alpha}) \leq \alpha\phi(x-\epsilon) + (1-\alpha)\phi(\frac{\alpha\epsilon}{1-\alpha}) \to \alpha\phi(x)$ as $\epsilon \downarrow 0$ by the continuity of ϕ and the assumption $\lim_{0+} \phi = 0$.

(ii.) Now by Jensen $\int \phi^- \circ (\alpha X) d\mu < \infty$, $\int \phi^- \circ X d\mu < \infty$, $\int X d\mu \in (0,\infty)$ and $\alpha \int \phi \circ X d\mu \geq \int \phi \circ (\alpha X) d\mu \geq \phi(\int \alpha X d\mu) = \phi(\alpha \int X d\mu)$.