MEASURE THEORY — "MIDTERM" EXAM

Time allowed: 90 min. Total no. of points: 100.
You may write in pen or pencil.

November 28th 2018

2^X je potenčna množica X. Leb je Lebesgueova mera na $\mathbb{R}$. $\mathcal{A}/\mathcal{B}$ je množica $\mathcal{A}/\mathcal{B}$-merljivih preslikav. $\mathcal{B}_X$ je Borelova σ-algebra na X pod standardno topologijo za X. $\overline{\mathcal{A}}^\mu$ je napolnitev σ-algebre $\mathcal{A}$ glede na mero μ na $\mathcal{A}$. 2^X *is the power set of* X. Leb *is the Lebesgue measure on* $\mathbb{R}$. $\mathcal{A}/\mathcal{B}$ *is the set of* $\mathcal{A}/\mathcal{B}$*-measurable maps.* $\mathcal{B}_X$ *is the Borel* σ*-algebra on* X *for the standard topology on* X. $\overline{\mathcal{A}}^\mu$ *is the completion of a* σ*-field* $\mathcal{A}$ *with respect to a measure* μ *on* $\mathcal{A}$.

1. Naj bo X množica, $h : X \to X$ bijekcija. Definirajmo $\mathcal{F}^\wedge := \{A \in 2^X : h(A) \subset A \text{ in } h^{-1}(A) \subset A\}$ in $\mathcal{F}^\vee := \{A \in 2^X : h(A) \subset A \text{ ali } h^{-1}(A) \subset A\}$. *Let* X *be a set,* $h : X \to X$ *a bijection. Set* $\mathcal{F}^\wedge := \{A \in 2^X : h(A) \subset A \text{ and } h^{-1}(A) \subset A\}$ *and* $\mathcal{F}^\vee := \{A \in 2^X : h(A) \subset A \text{ or } h^{-1}(A) \subset A\}$.
   (a) (14 points) Ali je $\mathcal{F}^\wedge$ σ-algebra na X, ne glede na izbiro h? Dokaži ali najdi protiprimer. *Is* $\mathcal{F}^\wedge$ *a* σ*-field on* X*, no matter what the* h*? Prove or find a counterexample.*
   (b) (14 points) Ali je $\mathcal{F}^\vee$ σ-algebra na X, ne glede na izbiro h? Dokaži ali najdi protiprimer. *Is* $\mathcal{F}^\vee$ *a* σ*-field on* X*, no matter what the* h*? Prove or find a counterexample.*
   (c) (12 points) Če je odgovor na del (a) pritrdilen, potem identificiraj, kolikor eksplicitno moreš, $\mathcal{F}^\wedge/\mathcal{B}_{[-\infty,\infty]}$. Če je odgovor na del (b) pritrdilen, potem identificiraj, kolikor eksplicitno moreš, $\mathcal{F}^\vee/\mathcal{B}_{[-\infty,\infty]}$. *Provided the answer to part* (a) *is to the affirmative, then identify, as explicitly as you can,* $\mathcal{F}^\wedge/\mathcal{B}_{[-\infty,\infty]}$*. Provided the answer to part* (b) *is to the affirmative, then identify, as explicitly as you can,* $\mathcal{F}^\vee/\mathcal{B}_{[-\infty,\infty]}$.

Solution: Clearly (o) $\mathcal{F}^\wedge \subset 2^X$ and (i) $X \in \mathcal{F}^\wedge$. Further, (ii) if $A \in \mathcal{F}^\wedge$, then $h(A) \subset A$, and hence $h^{-1}(X\backslash A) \subset X\backslash A$, similarly $h^{-1}(A) \subset A$, and hence $h(X\backslash A) \subset X\backslash A$. It follows that $X\backslash A \in \mathcal{F}^\wedge$. Finally, (iii) $\mathcal{F}^\wedge$ is closed under countable unions, sicne $h(\cup_{i\in\mathbb{N}}A_i) = \cup_{i\in\mathbb{N}}h(A_i)$ and likewise $h^{-1}(\cup_{i\in\mathbb{N}}A_i) = \cup_{i\in\mathbb{N}}h^{-1}(A_i)$, whenever $(A_i)_{i\in\mathbb{N}}$ is a sequence in 2^X. It follows that $\mathcal{F}^\wedge$ is a σ-field on X. The answer for $\mathcal{F}^\vee$ is negative: it is clear from the above it verifies (o)-(i)-(ii), but, in general not (iii): for instance with $h = \mathrm{id}_\mathbb{R}/2$, $A_1 := \{2^{-n} : n \in \mathbb{N}\} \in \mathcal{F}^\vee$ and $A_2 := \{2^n : n \in \mathbb{N}\} \in \mathcal{F}^\vee$, yet $A_1 \cup A_2 \notin \mathcal{F}^\vee$ (as, e.g., $\{1/2, 2\} \subset A_1 \cup A_2$, $1 \notin A_1 \cup A_2$). We wish lastly to identify $\mathcal{F}^\wedge/\mathcal{B}_{[-\infty,\infty]}$. If $A \in \mathcal{F}^\wedge$, then $\mathbb{1}_A \circ h = \mathbb{1}_{h^{-1}(A)} \leq \mathbb{1}_A$ and similarly $\mathbb{1}_A \circ h^{-1} \leq \mathbb{1}_A$. Hence $\mathbb{1}_A = \mathbb{1}_A \circ h$. (Incidentally, of course $\mathcal{F}^\wedge = \{A \in 2^X : h(A) = A = h^{-1}(A)\} = \{A \in 2^X : A = h^{-1}(A)\} = \{A \in 2^X : h(A) = A\}$.) For instance from the usual arguments of linearity and approximation, it then follows that for all $f \in \mathcal{F}^\wedge/\mathcal{B}_{[-\infty,\infty]}$, one has $f = f \circ h$. Conversely, if for some $f \in [-\infty,\infty]^X$ one has $f = f \circ h$, whence $f = f \circ h^{-1}$, then for $R \in \mathcal{B}_{[-\infty,\infty]}$, $f^{-1}(R) = h(f^{-1}(R)) = h^{-1}(f^{-1}(R))$, and thus indeed $f^{-1}(R) \in \mathcal{F}^\wedge$. In conclusion:

$$\mathcal{F}^\wedge/\mathcal{B}_{[-\infty,\infty]} = \{f \in [-\infty,\infty]^X : f = f \circ h\}.$$

Remark: It is easy to see that if X is finite, then $\mathcal{F}^\vee = \mathcal{F}^\wedge$. If X is infinite, then the example of h given in part (b) can be easily (since the bijection h given there fixes $\{2^k : k \in \mathbb{Z}\}$) tweaked to show that there is some bijection h *on the given* X for which $\mathcal{F}^\vee$ is not a σ-field on X. Finally, in part (c), in fact for any separated σ-field $\mathcal{H}$ one has

$$\mathcal{F}^\wedge/\mathcal{H} = \{f \in (\cup\mathcal{H})^X : f = f \circ h\}.$$

The inclusion $\supset$ is proved in precisely the same way, whilst the inclusion $\subset$ follows from the separatedness: if there is $x \in X$ with $f(x) \neq f(h(x))$, there is an $H \in \mathcal{H}$ with $\mathbb{1}_H(f(x)) \neq \mathbb{1}_H(f(h(x)))$, which implies $h^{-1}(f^{-1}(H)) \neq f^{-1}(H)$, and so $f^{-1}(H)$ does not belong to $\mathcal{F}^\wedge$. [[By definition $\mathcal{H}$ is separated if given any $x \neq y$ from $\cup\mathcal{H}$, there is $H \in \mathcal{H}$ such that $\mathbb{1}_H(x) \neq \mathbb{1}_H(y)$.]] If $\mathcal{H}$ is not separated, then in general the displayed equality fails; for instance if $\mathcal{H}$ is the trivial σ-field on a space containing at least two points and h is not the identity.

2. (30 points) Naj bo $A \subset \mathbb{R}$. Dokaži, da sta sledeči trditvi ekvivalentni: *Let* $A \subset \mathbb{R}$. *Prove that the following two statements are equivalent:*

   (i) A je Lebesguovo merljiva. A *is Lebesgue measurable.*

   (ii) Obstajajo odprte $O_n \subset \mathbb{R}$ in zaprte $K_n \subset \mathbb{R}$, $n \in \mathbb{N}$, da je $\cup_{n\in\mathbb{N}}K_n \subset A \subset \cap_{n\in\mathbb{N}}O_n$ in $\mathrm{Leb}((\cap_{n\in\mathbb{N}}O_n)\backslash(\cup_{n\in\mathbb{N}}K_n)) = 0$. *There exist open* $O_n \subset \mathbb{R}$ *and closed* $K_n \subset \mathbb{R}$, $n \in \mathbb{N}$, *such that* $\cup_{n\in\mathbb{N}}K_n \subset A \subset \cap_{n\in\mathbb{N}}O_n$ *and* $\mathrm{Leb}((\cap_{n\in\mathbb{N}}O_n)\backslash(\cup_{n\in\mathbb{N}}K_n)) = 0$.

   Sklepaj, da je $\mathcal{L} = \overline{\mathcal{B}_\mathbb{R}}^{\mathrm{Leb}|_{\mathcal{B}_\mathbb{R}}}$, kjer je $\mathcal{L}$ množica Lebesguovo merljivih množic. *Conclude that* $\mathcal{L} = \overline{\mathcal{B}_\mathbb{R}}^{\mathrm{Leb}|_{\mathcal{B}_\mathbb{R}}}$, *where* $\mathcal{L}$ *is the collection of Lebesgue measurable sets.*

Solution: $\downarrow$ implication: The Lebesgue measure is outer regular wrt open sets and inner regular wrt closed (or compact) sets. We use the following known consequence thereof: for any $\epsilon > 0$ there are K and O closed and open respectively, $K \subset A \subset O$ with $\mathrm{Leb}(O\backslash K) < \epsilon$. Then we find for each $n \in \mathbb{N}$, K_n and O_n, closed and open respectively, with $K_n \subset A \subset O_n$, and such that $\mathrm{Leb}(O_n\backslash K_n) < 1/2^n$. Then $\cup_{n\in\mathbb{N}}K_n \subset A \subset \cap_{n\in\mathbb{N}}O_n$ and by monotonicity, for each $m \in \mathbb{N}$, $\mathrm{Leb}((\cap_{n\in\mathbb{N}}O_n)\backslash(\cup_{n\in\mathbb{N}}K_n)) \leq \mathrm{Leb}(O_m\backslash K_m) < 1/2^m$ which $\downarrow 0$ as $m \to \infty$. [[In particular we see that the Lebesgue σ-field $\mathcal{L}$ is contained in the completion of the Borel σ-field with respect to the restriction of the Lebesgue measure to the Borel σ-field. Since the Lebesgue measure is complete, it means that one has in fact the equality $\mathcal{L} = \overline{\mathcal{B}_\mathbb{R}}^{\mathrm{Leb}|_{\mathcal{B}_\mathbb{R}}}$.]] The implication $\uparrow$ is clear: A differs from a Borel measurable set only by a subset of a negligible Borel set, and is hence Lebesgue measurable (the Lebesgue measure is complete).

3. (30 points) Izračunaj: *Compute*:

$$\lim_{M\to\infty}\lim_{n\to\infty} n^3 \int_{\frac{1}{n^2}}^{\frac{M}{n}} (1-x)^n x^2 \cos(x)dx$$

(limite po n, M čez $\mathbb{N}$ *limits of* n, M *over* $\mathbb{N}$).

Solution: Forgetting about the outer limit for the time being, we effect the changes of variables $u = xn$ in order to obtain

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^{M} (1 - u/n)^n u^2 \cos(u/n) du.$$

We then pass to Lebesgue integrals via equality of Riemann and Lebesgue integrals of continuous functions on compact intervals:

$$\lim_{n\to\infty} \int \mathbb{1}_{[\frac{1}{n},M]}(u)(1 - u/n)^n u^2 \cos(u/n)\mathrm{Leb}(du)$$

followed by an application of monotone convergence (cos is decreasing and nonnegative on $[0, \pi/2]$: no matter what the M, for all sufficiently large n, we are "home safe"; also, for any fixed $x \in (0, \infty)$, the map $[x, \infty) \ni \alpha \mapsto \left(1 - \frac{x}{\alpha}\right)^\alpha \in [0, \infty)$ is nondecreasing to e^{-x} as is well-known and may be checked e.g. via differentiation: again no matter how large the M eventually in n we are good) to get $\int \mathbb{1}_{(0,M]}(u)e^{-u}u^2\mathrm{Leb}(du) = \int \mathbb{1}_{[0,M]}(u)e^{-u}u^2\mathrm{Leb}(du)$. In the limit as $M \to \infty$ we obtain the (improper) Riemann integral: $\int_0^\infty u^2 e^{-u} du$, which is (via a double per partes or otherwise) $= \Gamma(3) = 2! = 2$.