MEASURE THEORY — "END-OF-TERM" EXAM

Time allowed: 120 min. Total no. of points: 110 (10 bunus points).
You may write in pen or pencil.

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$A \subset B \overset{\text{def}}{\Leftrightarrow} \forall x (x \in A \Rightarrow x \in B)$. 2^X je potenčna množica X. m je Lebesgueova mera na $\mathbb{R}$. $\mathcal{A}/\mathcal{B}$ je množica $\mathcal{A}/\mathcal{B}$-merljivih preslikav. $\mathcal{B}_X$ je Borelova σ-algebra na X pod standardno topologijo za X. $\overline{\mathcal{A}}^\mu$ je napolnitev σ-algebre $\mathcal{A}$ glede na mero μ na $\mathcal{A}$. $A \subset B \overset{\text{def}}{\Leftrightarrow} \forall x (x \in A \Rightarrow x \in B)$. *$2^X$ is the power set of X. m is the Lebesgue measure on $\mathbb{R}$. $\mathcal{A}/\mathcal{B}$ is the set of $\mathcal{A}/\mathcal{B}$-measurable maps. $\mathcal{B}_X$ is the Borel σ-algebra on X for the standard topology on X. $\overline{\mathcal{A}}^\mu$ is the completion of a σ-field $\mathcal{A}$ with respect to a measure μ on $\mathcal{A}$.*

1. (25 points) Za $\{p, q\} \subset (0, \infty)$ izračunaj $\int_0^1 \frac{x^{p-1}}{1+x^q} dx$ (izrazi kot numerično vrsto). Kateri (znani) vrsti dobiš pri $p = q = 1$ ter pri $p = 1$, $q = 2$? **Nasvet**: Integrand razvij v, in integriraj funkcijsko vrsto. *For $\{p, q\} \subset (0, \infty)$ evaluate $\int_0^1 \frac{x^{p-1}}{1+x^q} dx$ (express as a numeric series). Which (known) series do you obtain with $p = q = 1$ and with $p = 1$, $q = 2$?* **Advice**: Expand the integrand into, and integrate a function series.

Solution: The integrand is continuous, we are living on a compact interval; the Lebesgue integral may replace the Riemann integral. Expanding into a series, for $x \in [0, 1)$, $\frac{x^{p-1}}{1+x^q} = x^{p-1}(1 - x^q + x^{2q} - x^{3q} + \cdots)$: The series is alternating; still we may "integrate term-by-term", if we combine two subsequent terms, since such sums are nonnegative (alternatively can appeal to dominated (bounded) convergence for the partial sums). We obtain $\int_0^1 \sum_{k=0}^\infty (x^{p-1+2qk} - x^{p-1+q(2k+1)}) dx = \sum_{k=0}^\infty \int_0^1 (x^{p-1+2qk} - x^{p-1+q(2k+1)}) dx = \sum_{k=0}^\infty \frac{1}{p+2qk} - \frac{1}{p+q(2k+1)} = \sum_{k=0}^\infty \frac{q}{(p+2qk)(p+q(2k+1))}$. With $p = q = 1$ we get the/a series for $\log 2$; with $p = 1$, $q = 2$, one gets the/a series for $\frac{\pi}{4}$.

2. Naj bo μ mera na $((0, \infty), \mathcal{B}_{(0,\infty)})$. Naj bo $f : (0, \infty) \to [0, \infty)$ odvedljiva, nepadajoča z $\lim_{0+} f = 0$. Pokaži, da je

$$\int f d\mu = \int_{(0,\infty)} f'(y) \mu((y, \infty)) m(dy)$$

Let μ be a measure on $((0, \infty), \mathcal{B}_{(0,\infty)})$. Let $f : (0, \infty) \to [0, \infty)$ be differentiable, nondecreasing with $\lim_{0+} f = 0$. Show that

$$\int f d\mu = \int_{(0,\infty)} f'(y) \mu((y, \infty)) m(dy)$$

(a) (25 points) če je μ σ-končna; *if μ is σ-finite;*

(b) (10 points) in še v splošnem. *and in general.*

Pomoč. Pod navedenimi pogoji je za realne $0 < a \leq b$, $f(b) - f(a) = \int_{[a,b]} f' dm$. ***Help.*** *Under the indicated conditions, for real $0 < a \leq b$, $f(b) - f(a) = \int_{[a,b]} f' dm$.*

Solution: By the nondecreasingness of f, $f' \geq 0$; also f' is Borel measurable (as a suitable limit ...). When μ is σ-finite one concludes via the Fundamental theorem of calculus coupled with the theorem of Tonelli, writing $f(x) = \int_{(0,x)} f' dm$, $x \in (0,\infty)$ (where we have taken into account $\lim_{0+} f = 0$, via monotone convergence) and noting that $((0,\infty) \times \mathbb{R} \ni (x,y) \mapsto \mathbb{1}_{(0,x)}(y) f'(y)) \in \mathcal{B}_{(0,\infty)} \otimes \mathcal{B}_{\mathbb{R}}/\mathcal{B}_{[0,\infty)} = \mathcal{B}_{(0,\infty)\times\mathbb{R}}/\mathcal{B}_{[0,\infty)}$. If μ is not σ-finite, then for some $y \in (0,\infty)$, $\mu(y,\infty) = \infty$. Let $y^* := \sup\{y \in (0,\infty) : \mu(y,\infty) = \infty\} > 0$. The case $y^* = \infty$ may be dealt with easily seperately (both sides of the target equality are 0 or ∞ according as to whether $f' = 0$ m-a.e. (and hence everywhere) or not); assume $y^* < \infty$. Notice that $\mathbb{1}_{(y^*,\infty)} \cdot \mu$ is σ-finite and hence what we have already proven applies. Also, if $f(y^*) = 0$, then $\int_{(0,y^*]} f d\mu = 0 = \int_0^{y^*} f'(y)\mu(y,\infty)dy$, and linearity of the integral allows to conclude. And if $f(y^*) > 0$, then: (i) by the continuity of f, f is bounded below away from zero on (z,∞) for some $z \in (0,y^*)$; whereas (ii) by the Fundamental theorem of calculus and since $\lim_{0+} f = 0$, $m(\{f' > 0\} \cap (0,y^*)) > 0$; whence (noting that $\mu(y,\infty) = \infty$ for all $y \in (0,y^*)$) $\int f d\mu = \infty = \int_{(0,\infty)} f(y)\mu(y,\infty)m(dy)$.

3. (25 points) Naj bo $(\mu_n)_{n\in\mathbb{N}_0}$ zaporedje kompleksnih mer na $(X,\mathcal{A})$. Pokaži, da sta sledeči trditvi ekvivalentni: *Let $(\mu_n)_{n\in\mathbb{N}_0}$ be a sequence of complex measures on $(X,\mathcal{A})$. Show that the following statements are equivalent:*

   (i) $\mu_n(A) \to \mu_0(A)$, ko gre $n \to \infty$, enakomerno v $A \in \mathcal{A}$. *$\mu_n(A) \to \mu_0(A)$, as $n \to \infty$, uniformly in $A \in \mathcal{A}$.*

   (ii) $\lim_{n\to\infty} \|\mu_n - \mu_0\| = 0$.

   Spomnimo/povejmo: Za kompleksno mero ν na $(X,\mathcal{A})$ je $\|\nu\| := |\nu|(X)$. ***Recall/be advised:*** *For a complex measure ν on $(X,\mathcal{A})$, $\|\nu\| := |\nu|(X)$.*

Solution: The first statement means, by definition, that $\lim_{n\to\infty} \sup_{A\in\mathcal{A}} |\mu_n(A) - \mu_0(A)| = 0$, i.e. $\lim_{n\to\infty} \sup_{A\in\mathcal{A}} |(\mu_n - \mu_0)(A)| = 0$. Since complex measures constitute a vector space over $\mathbb{C}$, it is then sufficient to show that for a sequence of complex measures $(\nu_n)_{n\in\mathbb{N}}$ on $(X,\mathcal{A})$, the statements

(I) $\nu_n(A) \to 0$, as $n \to \infty$, uniformly in $A \in \mathcal{A}$.

(II) $\lim_{n\to\infty} \|\nu_n\| = 0$.

are equivalent. Sufficiency of the condition follows from $|\nu(A)| \leq |\nu|(A) \leq |\nu|(X) = \|\nu\|$ (where ν is a complex measure ...). Necessity. For $n \in \mathbb{N}$ let rP_n and rN_n, respectively iP_n and iN_n, be a Hahn decomposition of the signed measure $\Re\nu_n$, respectively $\Im\nu_n$. Then for any measurable partition $(A_j)_{j=1}^k$ of X and any $n \in \mathbb{N}$, $\sum_{j=1}^k |\nu_n(A_j)| \leq |\nu_n(rP_n)| + |\nu_n(rN_n)| + |\nu_n(iP_n)| + |\nu_n(iN_n)|$. Taking the supremum over the partitions, and sending $n \to \infty$, yields the desired conclusion.

4. (25 points) Naj bo $(X,\mathcal{F},\mu)$ prostor z mero, $\mu(X) = 1$, $f \in \mathcal{F}/\mathcal{B}_{(e^{-1},\infty)}$, $\int f d\mu = e$. Dokaži, da je $\int \sqrt{1+\ln(f)} d\mu \leq \sqrt{2}$. *Let $(X,\mathcal{F},\mu)$ be a measure space, $\mu(X) = 1$, $f \in \mathcal{F}/\mathcal{B}_{(e^{-1},\infty)}$, $\int f d\mu = e$. Prove that $\int \sqrt{1+\ln(f)} d\mu \leq \sqrt{2}$.*

Solution: Differentiating $\phi := ((e^{-1}, \infty) \ni x \mapsto \sqrt{1 + \ln(x)})$ twice yields $\phi'' < 0$ which implies ϕ is concave. Then Jensen.