Combinatorics: 1st exam

29 January 2018

The exam lasts for 120 minutes. You can achieve 100 points. Good luck!

Problem 1 (20 points)

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. Prove the following:

a) $\displaystyle\sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda_i'}{2}$,

b) $\displaystyle\sum_{i\geq 1}\left\lfloor \frac{\lambda_{2i-1}}{2}\right\rfloor = \sum_{i\geq 1}\left\lceil \frac{\lambda_{2i}'}{2}\right\rceil$.

Problem 2 (25 points)

Suppose that $2n$ people are sitting in a circle. Note that people have names.

a) Let $c_{n,k}$ denote the number of ways we can choose k pairs of adjacent people among them. Show that

$$c_{n,k} = \binom{2n-k-1}{k-1} + \binom{2n-k}{k}.$$

b) Suppose we have k fixed pairs of adjacent people. Let $a_{n,k}$ denote the number of ways we can form $n-k$ pairs between the remaining $2n-2k$ people (with no restriction on the adjacency). Show that

$$a_{n,k} = (2n-2k-1)!!.$$

c) Determine the number of ways these $2n$ people can form n pairs if no two adjacent people can be in the same pair. Express your answer as a finite sum.

Problem 3 (30 points)

Consider a_n, the number of paths between the points $(0,0)$ and (n,n) with steps $(0,1)$, $(1,0)$, and $(1,1)$.

a) Prove that

$$a_n = \sum_k \binom{n+k}{k,k,n-k}.$$

b) Using the above relation and equalities

$$\frac{1}{(1-x)^{2k+1}} = \sum_n \binom{n+2k}{2k}x^n, \qquad \frac{1}{\sqrt{1-4x}} = \sum_k \binom{2k}{k}x^k,$$

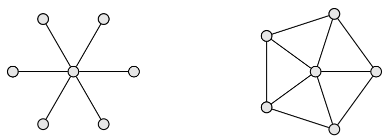
find an explicit formula for $\sum_n a_n x^n$.

c) Determine the asymptotic behavior of the sequence $(a_n)_n$.

Problem 4 (25 points)

Let d_n denote the number of labeled graphs on n vertices which have connected components isomorphic to a star or a wheel. For example, $d_0 = 1$, $d_1 = d_2 = d_3 = 0$, $d_4 = 8$, $d_5 = 20$, and $d_6 = 78$.

Remember that a star $K_{1,n}$, $n \geq 3$, is a graph with $n+1$ vertices. One of them has n neighbors, all others have just one. A wheel W_n, $n \geq 3$, is a graph with $n+1$ vertices, n of them form a cycle, and one (central) vertex is connected to all vertices on this cycle. The figure below shows unlabeled graphs $K_{1,6}$ and W_5.



a) Determine the exponential generating function for the sequence $(d_n)_n$.

b) Let $F(x) = \sum_n a_n \frac{x^n}{n!}$, $H(x) = \sum_n b_n \frac{x^n}{n!}$ and $H(x) = e^{F(x)}$. Prove that for all $n \in \mathbb{N}$ it holds

$$nb_n = \sum_k \binom{n}{k} k a_k b_{n-k}.$$

Hint: What is the exponential generating function for the sequence $(nb_n)_n$*?*

c) Can you get a recurrence equation for $(d_n)_n$?