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Combinatorics: 2nd exam

19 April 2018

The exam lasts for 120 minutes. You can achieve 100 points. Good luck!

Name and surname

Student ID

Problem 1 (20 points)

Let $n - l = 2k$ and let $f(n, l, s)$ be the number of compositions of n with l odd parts and s even parts. Give a bijective proof that $f(n, l, s) = \binom{l+s}{l}\binom{l+k-1}{l+s-1}$.

Problem 2 (25 points)

Say that a permutation $\pi \in S_{2n}$ has property $\mathcal{P}$ if for some $i \in [2n]$, $|\pi(i) - \pi(i+1)| = n$, where $i+1$ is taken modulo $2n$. Show that, for each n, there are more permutations with property $\mathcal{P}$ than without it.

Hint: You can use the inequality

$$\left|\bigcup_{i=1}^{m} A_i\right| \geq \sum_{i=1}^{m} |A_i| - \sum_{\{i,j\} \subset [m]} |A_i \cap A_j|.$$

Problem 3 (30 points)

A sequence $(a_n)_n$ is defined as $a_0 = 1$, $a_1 = 3$ and

$$a_n = 2a_{n-1} + a_{n-2}.$$

a) Find the ordinary generating function $A(x) = \sum_{n \geq 0} a_n x^n$.

b) Find singularities of $A(x)$ and discuss the asymptotic behavior of the sequence $(a_n)_n$.

c) Determine a_n.

Problem 4 (25 points)

Let d_n denote the number of ways a group of n children can separate into an odd number of subgroups, where each subgroup consists of at least three children, who stand in a circle, and all other children in this group stand behind one of the children in the circle. Note that the circle always consists only of three children. For example, $d_0 = 1$, $d_1 = d_2 = 0$, $d_3 = 2$, $d_4 = 24$.

Determine the exponential generating function $D(x) = \sum_n d_n \frac{x^n}{n!}$.

Example (without names of the 14 children):