Logika in množice: 2. kolokvij

14. januar 2010

Čas reševanja je 120 minut. Doseženih 100 točk šteje za maksimalno oceno.

1. naloga (30 točk)

Na množici $\mathbb{N}$ je dana relacija $R \subseteq \mathbb{N} \times \mathbb{N}$, definirana s predpisom

$$m \; R \; n \iff \exists\, k \in \mathbb{N} \,.\, 2^k \cdot m = n.$$

Ugotovi, katere od naslednjih lastnosti ima R:

a) refleksivna,

b) irefleksivna,

c) simetrična,

d) tranzitivna,

e) antisimetrična,

f) funkcijska.

2. naloga (25 točk)

Na množici $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ definiramo ekvivalenčno relacijo $\sim$ s predpisom

$$(a, b) \sim (c, d) \iff a \cdot d = b \cdot c.$$

Naštej ekvivalenčne razrede relacije $\sim$.

3. naloga (25 točk)

Potenčna množica realnih števil $\mathcal{P}(\mathbb{R})$ je polna mreža glede na relacijo podmnožica $\subseteq$. Naj bo $S \subseteq \mathcal{P}(\mathbb{R})$ množica

$$S = \{(-a, a) \subseteq \mathbb{R} \mid a \in \mathbb{R} \wedge a > 1\},$$

kjer je $(-a, a) = \{x \in \mathbb{R} \mid -a < x < a\}$ odprti interval s krajiščema $-a$ in a.

a) Ali S ima infimum v $(\mathcal{P}(\mathbb{R}), \subseteq)$? Če ga ima, ga izračunaj.

b) Ali S ima supremum v $(\mathcal{P}(\mathbb{R}), \subseteq)$? Če ga ima, ga izračunaj.

4. naloga (40 točk)

a) Ali je množica $S = \left\{f \in \mathbb{N}^{\mathbb{N}} \;\middle|\; \forall\, n \in \mathbb{N} \,.\, f(n) \leq f(n+1)\right\}$ števna?

b) Ali je množica $T = \left\{f \in \mathbb{N}^{\mathbb{N}} \;\middle|\; \forall\, n \in \mathbb{N} \,.\, f(n) \geq f(n+1)\right\}$ števna?

5. naloga (za čast in slavo)

Poišči kako neštevno verigo v polni mreži $(\mathcal{P}(\mathbb{Q}), \subseteq)$.

Logic and sets: Second midterm exam

19 January 2010

Available time is 120 minutes, 100 points are worth the maximum grade.

Question 1 (25 marks)

Define the relation $S \subseteq \mathbb{R} \times \mathbb{R}$ on the real numbers $\mathbb{R}$ by

$$x \; S \; y \iff x - y \in \mathbb{Q}.$$

In words, x and y are related by S when $x - y$ is a rational number. Determine which of the following properties S has:

a) reflexive,

b) irreflexive,

c) symmetric,

d) antisymmetric,

e) transitive.

f) (+10 marks) Let T be the least equivalence relation that contains S. Is the quotient set $\mathbb{R}/T$ countable?

Question 2 (25 marks)

a) We equip the set $S = \{1, 2, \ldots, 16\}$ with relation $\preceq$ defined by

$$m \preceq n \iff \exists\, k \in \mathbb{N} \,.\, m^k = n.$$

Draw a *beautiful and clear* Hasse diagram of the partial order $(S, \preceq)$.

b) (+10 marks) Does the supremum $m \vee n$ exist for all $m, n \in S$? How about the infimum $m \wedge n$?

Question 3 (25 marks)

Define the relation $\sim$ on the set of real numbers $\mathbb{R}$ by

$$x \sim y \iff \cos(x - y) = 1.$$

a) Verify that $\sim$ is an equivalence relation.

b) We would like to define operations $\oplus$ and $\otimes$ on the quotient set $\mathbb{R}/{\sim}$ with the rules

$$[x] \oplus [y] = [x + y],$$
$$[x] \otimes [y] = [x \cdot y].$$

Are $\oplus$ and $\otimes$ well defined?

Question 4 (25 marks)

Let $(P, <_P)$ and $(Q, <_Q)$ be well-ordered sets. We define the relation $\sqsubset$ on the cartesian product $P \times Q$ by

$$(u, v) \sqsubset (x, y) \iff u <_P x \vee (u = x \wedge v <_Q y).$$

Relation $\sqsubset$ is a linear order, which you do not have to prove. Prove that $(P \times Q, \sqsubset)$ is a well-ordered set.

Question 5 (honor and fame)

A cardinal number κ is *regular*, if it cannot be obtained as a sum of fewer than κ cardinal numbers which are all smaller than κ. More precisely, if $\{S_i \mid i \in I\}$ is a family of cardinal numbers such that $|I| < \kappa$ and $|S_i| < \kappa$ for all $i \in I$, then $|\coprod_{i \in I} S_i| < \kappa$.

a) (honor) Find a regular cardinal number.

b) (a little bit of fame) Find two regular cardinal numbers.

c) (a lot of fame) Find three regular cardinal numbers.

Logic and sets: Second midterm exam

28 January 2010

Available time is 120 minutes, 100 points are worth the maximum grade.

Question 1 (30 marks)

Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be the relation on $\mathbb{N}$ defined by

$$m \, R \, n \iff \exists \, k \in \mathbb{N} \, . \, 2^k \cdot m = n.$$

Determine which of the following properties R has:

a) reflexive,

b) irreflexive,

c) symmetric,

d) transitive,

e) antisymmetric,

f) functional.

Question 2 (25 marks)

We define the relation $\sim$ on the set $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ by

$$(a, b) \sim (c, d) \iff a \cdot d = b \cdot c.$$

List the equivalence classes of $\sim$.

Question 3 (25 marks)

The powerset of real numbers $\mathcal{P}(\mathbb{R})$ is a complete lattice with respect to the subset relation $\subseteq$. Let $S \subseteq \mathcal{P}(\mathbb{R})$ be the set

$$S = \{(-a, a) \subseteq \mathbb{R} \mid a \in \mathbb{R} \wedge a > 1\},$$

where $(-a, a) = \{x \in \mathbb{R} \mid -a < x < a\}$ is the open interval with endpoints $-a$ and a.

a) Does S have the infimum in $(\mathcal{P}(\mathbb{R}), \subseteq)$? If it does, compute it.

b) Does S have the supremum in $(\mathcal{P}(\mathbb{R}), \subseteq)$? If it does, compute it.

Question 4 (40 marks)

a) Is the set $S = \{f \in \mathbb{N}^{\mathbb{N}} \mid \forall n \in \mathbb{N} \, . \, f(n) \leq f(n+1)\}$ countable?

b) Is the set $T = \{f \in \mathbb{N}^{\mathbb{N}} \mid \forall n \in \mathbb{N} \, . \, f(n) \geq f(n+1)\}$ countable?

Question 5 (za čast in slavo)

Find an uncountable chain in the complete lattice $(\mathcal{P}(\mathbb{Q}), \subseteq)$.