Logika in množice: 1. kolokvij

11. december 2013

Čas reševanja je 120 minut. Vse odgovore utemeljite. Veliko uspeha!

1. naloga (25 točk)

Krajevna skupnost Domena Interpretacije[1] ima 100 prebivalcev, razdeljenih v tri vasi,[2] ki se imenujejo Hrib, Reka in Jezero.

Radi bi se pogovarjali o teh prebivalcih, zato vpeljemo enomestne predikatne simbole H, R, J, C, ki naj po vrsti izražajo lastnosti „je prebivalec Hriba“, „je prebivalec Reke“, „je prebivalec Jezera“ in „nosi čevlje“; ter dvomestni predikatni simbol P, pri čemer naj $P(x, y)$ pomeni „x pozna y“.

Naslednje trditve o prebivalcih Domene Interpretacije prevedi v jezik predikatnega računa:

a) Noben prebivalec Jezera ne nosi čevljev.

b) Vsak prebivalec Hriba pozna kakega prebivalca Reke.

c) Nekega prebivalca Hriba poznajo vsi prebivalci Reke.

d) Nekega prebivalca Reke, ki nosi čevlje, poznajo vsi prebivalci Jezera.

Recimo, da so vse trditve o prebivalcih Domene Interpretacije iz prejšnjih primerov resnične. Prevedi naslednje trditve v naravni jezik in ugotovi njihovo resničnost. (Dokazovanje ni potrebno.)

e) $\exists x : (J(x) \wedge \neg C(x))$

f) $\forall x : H(x) \Rightarrow \forall x : R(x)$

g) $\exists x \exists y : (C(x) \wedge H(y) \wedge P(x, y))$

2. naloga (30 točk)

Če je sklep veljaven, ga dokaži, sicer ga ovrzi s protiprimerom.

a) $p \vee q \vee r,\ q \vee r \vee s,\ r \vee s \vee t \ \models\ p \vee r \vee t$

b) $(p \wedge q) \vee (r \wedge s),\ \neg r \wedge \neg s \ \models\ p$

c) $p \Rightarrow (q \Rightarrow r),\ p \vee r,\ q \wedge s,\ r \Rightarrow (s \Rightarrow t) \ \models\ t$

3. naloga (25+5 točk)

Definirajmo tromestni logični veznik $I(p, q, r)$ kot

$$I(p, q, r) \equiv p \wedge (q \vee r)$$

a) Dokaži, da nabor $\{I\}$ ni poln.

b) Dokaži, da je nabor $\{1, \oplus, I\}$ poln.

c) S pomočjo veznikov 1, $\oplus$ in I izrazi veznik $\not\Rightarrow$, kjer je $p \not\Rightarrow q \equiv \neg(p \Rightarrow q)$.
Za dodatne točke: poišči čimkrajši zapis, ki naj čimmanjkrat uporabi spremenljivki p in q.

d) Poišči veznik J, tako da bo $\{I, J\}$ poln nabor in $\{J\}$ ne bo poln nabor.

[1] množica njenih prebivalcev naj bo tudi domena naše interpretacije

[2] vsaka vas ima vsaj enega prebivalca; vsak prebivalec je prebivalec natanko ene vasi

4. naloga (25 točk)

Dokaži sklep ali ga ovrzi s protiprimerom. Dokazov ni potrebno formalizirati.

a) $\forall x \forall y : (P(x) \wedge Q(y) \Leftrightarrow R(x,y)),\ \forall x \forall y : (R(x,y) \Rightarrow R(y,x)) \models \forall x : (P(x) \Leftrightarrow Q(x))$

b) $\forall x \forall y : (P(x) \wedge Q(y) \Leftrightarrow R(x,y)),\ \forall x \forall y : (R(x,y) \Rightarrow R(y,x))$
$\models \exists x : P(x) \wedge \exists x : Q(x) \Rightarrow \forall x : (P(x) \Leftrightarrow Q(x))$

Logic and Sets: First Midterm Exam

December 11, 2013

You have 120 minutes to complete your solutions. Justify your answers. Good luck!

Problem 1 (25 points)

The Domain of Interpretation local community[3] has 100 inhabitants, divided into three villages[4] called Hill, River and Lake.

In order to discuss these inhabitants, let H, R, L, S be unary predicate symbols which we interpret as „is an inhabitant of Hill“, „is an inhabitant of River“, „is an inhabitant of Lake“ and „wears shoes“. Let K be a binary predicate symbol, and interpret $K(x,y)$ as „x knows y“.

Translate the following statements about inhabitants of the Domain of Interpretation into the language of predicate calculus:

a) No inhabitant of Lake wears shoes.

b) Each inhabitant of Hill knows an inhabitant of River.

c) There is an inhabitant of Hill who is known to every inhabitant of River.

d) There is a certain inhabitant of River who wears shoes and is known to every inhabitant of Lake.

Suppose that all of the previous statements are true. Translate the following statements into natural language and determine which of them are true. (Proofs are not necessary.)

e) $\exists x : L(x) \wedge \neg S(x)$

f) $\forall x : H(x) \Rightarrow \forall x : R(x)$

g) $\exists x \exists y : (S(x) \wedge H(y) \wedge K(x,y))$

Problem 2 (30 points)

If the inference is valid, prove it, otherwise construct a counterexample.

a) $p \vee q \vee r,\ q \vee r \vee s,\ r \vee s \vee t\ \models\ p \vee r \vee t$

b) $(p \wedge q) \vee (r \wedge s),\ \neg r \wedge \neg s\ \models\ p$

c) $p \Rightarrow (q \Rightarrow r),\ p \vee r,\ q \wedge s,\ r \Rightarrow (s \Rightarrow t)\ \models\ t$

Problem 3 (25+5 points)

Define a logical connective $I(p,q,r)$ of arity 3 by

$$I(p,q,r) \equiv p \wedge (q \vee r)$$

a) Prove that $\{I\}$ is not a functionally complete set of connectives.

b) Prove that $\{1, \oplus, I\}$ is functionally complete.

c) Using 1, $\oplus$ and I express the connective $\not\Rightarrow$ defined by $p \not\Rightarrow q \equiv \neg(p \Rightarrow q)$.
For extra points: find an expression that is as short as possible, using p and q as few times as possible.

d) Find a connective J such that $\{I, J\}$ is functionally complete and $\{J\}$ is not.

[3]let the set of its inhabitants also be the domain of our interpretation

[4]each village has at least one inhabitant; each inhabitant lives in exactly one village

Problem 4 (25 points)

If the inference is valid, prove it, otherwise construct a counterexample. The proofs need not be formalized.

a) $\forall x \forall y : (P(x) \land Q(y) \Leftrightarrow R(x,y)),\ \forall x \forall y : (R(x,y) \Rightarrow R(y,x)) \models \forall x : (P(x) \Leftrightarrow Q(x))$

b) $\forall x \forall y : (P(x) \land Q(y) \Leftrightarrow R(x,y)),\ \forall x \forall y : (R(x,y) \Rightarrow R(y,x))$
$\models \exists x : P(x) \land \exists x : Q(x) \Rightarrow \forall x : (P(x) \Leftrightarrow Q(x))$