Logika in množice: 2. kolokvij

22. januar 2014

Čas reševanja je 120 minut. Vse odgovore utemeljite. Veliko uspeha!

1. naloga (30 točk)

Dani sta množici A in B. Ugotovi, kdaj je enačba v spremenljivki X rešljiva, in v tem primeru poišči rešitev.

$$X \cup A = X \oplus B$$

Rešitev enačbe čim bolj poenostavi.

2. naloga (30 točk)

Naj bo P_n množica vseh premic v n-razsežnem prostoru $\mathbb{R}^n$. Na množici P_n definiramo relacijo

$$p \perp_n q \overset{\text{def}}{\iff} \text{premica } p \text{ je pravokotna na premico } q.$$

Premici sta pravokotni, ko se sekata v neki točki in sta pravokotna njuna smerna vektorja.

a) Kateri znani relaciji je enaka relacija $(\perp_2)^2$?

b) Katere od relacij $\perp_2$, $(\perp_2)^2$ in $(\perp_2)^+$ so ekvivalenčne? Zanje opiši ekvivalenčne razrede!

c) Opiši relacijo $(\perp_3)^2$ (ni potrebno podrobno utemeljevati, zakaj je taka)! Ali je ekvivalenčna? Če je, kateri so njeni ekvivalenčni razredi?
Namig: različni premici v $\mathbb{R}^3$ sta bodisi vzporedni, bodisi mimobežni, bodisi se sekata.

3. naloga (25 točk)

Naj bo P množica vseh polinomov p z realnimi koeficienti in lastnostjo $p(0) = 0$ in naj bo $P_1 \subseteq P$ množica vseh tistih elementov množice P, ki so stopnje največ 1. Na množici P definiramo relacijo

$$p \preceq q \overset{\text{def}}{\iff} \exists C \in \mathbb{R} : \forall x \in \mathbb{R} : q(x) - p(x) \geq C.$$

Z istim predpisom definiramo relacijo $\preceq_1$ na množici P_1.

a) Dokaži, da je $\preceq$ relacija delne urejenosti.
Namig: preden se lotiš dokazovanja antisimetričnosti, se spomni, kateri polinomi imajo lastnost, da so hkrati navzgor in navzdol omejeni.

b) Iz prejšnje točke sledi, da je $\preceq_1$ tudi relacija delne urejenosti. (Tega ni treba dokazovati.) Dokaži, da je poleg tega $\preceq_1$ tudi simetrična. Kateri znani relaciji je enaka $\preceq_1$?

4. naloga (25 točk)

Naj bodo W, X, Y in Z množice in naj bodo $f : W \to X$, $g : X \to Y$ ter $h : Y \to Z$ preslikave.

a) Denimo, da sta $g \circ f$ in $h \circ g$ injektivni. Dokaži, da je $h \circ g \circ f$ injektivna. S protiprimerom pokaži, da h ni nujno injektivna.

b) Denimo, da sta $g \circ f$ in $h \circ g$ surjektivni. Dokaži, da je $h \circ g \circ f$ surjektivna. S protiprimerom pokaži, da f ni nujno surjektivna.

c) Denimo, da sta $g \circ f$ in $h \circ g$ bijektivni. Dokaži, da so h, g, f in $h \circ g \circ f$ bijektivne.

Logic and Sets: Second Midterm Exam

January 22, 2014

You have 120 minutes to complete your solutions. Justify your answers. Good luck!

Problem 1 (30 points)

Sets A and B are given. Determine under which conditions the following equation in X is solvable, and find the solution.

$$X \cup A = X \oplus B$$

Simplify your solution as much as possible.

Problem 2 (30 points)

Let P_n be the set of all lines in the n-dimensional space $\mathbb{R}^n$. We define the following relation on P_n:

$$p \perp_n q \overset{\text{def}}{\iff} \text{the line } p \text{ is perpendicular to the line } q.$$

Two lines are perpendicular if they intersect in a common point and their direction vectors are orthogonal.

a) Which known relation is $(\perp_2)^2$?

b) Which of the relations $\perp_2$, $(\perp_2)^2$ and $(\perp_2)^+$ are equivalence relations? Describe their equivalence classes!

c) Describe the relation $(\perp_3)^2$ (detailed justification of this description is not necessary)! Is it an equivalence relation? If so, what are its equivalence classes?

Hint: two distinct lines in $\mathbb{R}^3$ can be either parallel, passing or intersecting.

Problem 3 (25 points)

Let P be the set of all polynomials p with real coefficients satisfying the property $p(0) = 0$ and let $P_1 \subseteq P$ be the set of all those elements of P whose degrees are at most 1. We define the following relation on P:

$$p \le q \overset{\text{def}}{\iff} \exists C \in \mathbb{R} : \forall x \in \mathbb{R} : q(x) - p(x) \ge C.$$

By the same formula, we also define a relation $\le_1$ on P_1.

a) Prove that $\le$ is a partial order.

Hint: before proving antisymmetry, try to remember which polynomials have the property of being bounded from above and below.

b) The previous point implies that $\le_1$ is also a partial order. (Proving this is not necessary.) Prove that $\le_1$ is also symmetric. Which known relation is $\le_1$?

Problem 4 (25 points)

Let W, X, Y and Z be sets and let $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$ be mappings.

a) Suppose $g \circ f$ and $h \circ g$ are injective. Prove that $h \circ g \circ f$ is injective. Using a counterexample, show that h is not necessarily injective.

b) Suppose $g \circ f$ and $h \circ g$ are surjective. Prove that $h \circ g \circ f$ is surjective. Using a counterexample, show that f is not necessarily surjective.

c) Suppose $g \circ f$ and $h \circ g$ are bijective. Prove that h, g, f and $h \circ g \circ f$ are bijective.